CS498: Algorithmic Engineering
Lecture 1

Chandra Chekuri & Elfarouk Harb

University of Illinois Urbana-Champaign

01/20/2026

Outline

1. Course Logistics
   - Differences from CS374 and CS473
   - Content and Types of Projects in Class
   - Prerequisites
   - Grading
   - LLM Usage Policy
2. History of Linear Programming
3. Linear Programming: The Basics
4. The Engineer's Diet Dilemma
5. Interpreting and Debugging Gurobi Output

From Proofs to Solvers

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While standard algorithms courses focus on proving what is *computable*,
this course focuses on implementing what is *necessary*.

Relation to CS 374

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- Focuses on **Modelling**: Transforming Problem A to B so a solver for B can handle it.

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- **Example Concept:** `How to Solve This 3-SAT Instance (1M vars) in < 5s`

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- **CS 473** analyzes the *internal mathematics of the engine*.
- Advanced algorithmic techniques (example: randomization, flow, advanced dynamic programming).
- Focus on proving efficiency (run-time) and approximation gurantees (bounds).

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- **CS 498** teaches you how to *drive the car*.
- We treat powerful solvers, that researchers have spent decades working on, as black boxes to be mastered.
- Focus on modeling complex constraints rather than implementing the solver itself.
- We still explain the theory behind the solvers, but the focus is on basics of theory

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CS 498: ”Launch”

NP-Hardness is a worst-case warning, not a law of physics. Use SAT/SMT solvers to **crush real-world instances**. No more Grantees.

Modern Tooling Stack

We move beyond "pseudocode" to industrial-grade Python libraries used in Operations Research and Deep Learning.

- **Optimization:**
  `Gurobi`, `Pyomo` (Linear & Integer Programming)
- **Logic & Verification:**
  `Z3`, `PySAT` (SMT & SAT Solvers)
- **Differentiation:**
  `PyTorch` (Autograd & Neural Networks)

Course Comparison Matrix

Feature	CS 374 / 473	CS 498
Primary Goal	Proofs & Analysis	Models & Implementations
Hardness	Prove it's impossible in worst case	Use solvers to solve your instance anyway
Key Tools	Pencil, Paper, LaTeX	Gurobi, Z3, PyTorch, ...
Style	Purely Theoretical	Hybrid (Basics of Theory + Implementation)

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3. Coding

Python Literacy Check

```python
import numpy as np

A = np.array([[1, 2], [3, 4]])
b = np.array([5, 6])

# If you know what this does

x = np.linalg.solve(A, b)

# or can look it up quick
# ...you're Gucci.
```

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Weekly Homeworks

- **Groups of 2-4**.
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- After Parts I, II, III.
- In-class, short, individual quizzes.
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- **Algorithmic Engineering**.
- Build a system, implement a paper, or optimize a complex pipeline.
- Compare performance (speed/quality).

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X Bad:

“Here is the PDF of the homework, solve Problem 3 for me.”

Questions?

Ready to build?

> _

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Motzkin's Thesis (1936)

Listed only **42 papers** in all of history on linear inequalities. Today, there are tens of thousands per year.

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- 1946: The Air Force asks: “**Can you mechanize the planning process?**”
- He built a dynamic model of resources and activities, but something was missing.
- Dantzig realized he needed an **Explicit Objective Function** to optimize on top of his linear constraints.

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The Setup:

- George Dantzig spent WWII planning US Air Force logistics by hand.
- 1946: The Air Force asks: "**Can you mechanize the planning process?**"
- He built a dynamic model of resources and activities, but something was missing.
- Dantzig realized he needed an **Explicit Objective Function** to optimize on top of his linear constraints.
- But how to solve a system with thousands of linear constraints and linear objective? He needed help.

Act III: Meeting Von Neumann (Oct 1947)

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The Revelation

Von Neumann stands up: “**Oh—that!**”
He proceeds to lecture Dantzig for 90 minutes on **Duality** and **Geometry**.
Von Neumann had already derived the theory of LP while inventing Game Theory.

Act IV: The Mic Drop

Conference, 1948: Dantzig presents LP to a room of heavyweights.
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"But we all know the world is non-linear."

Dantzig freezes. The room goes silent. Then Von Neumann raises his hand:

***"If the axioms of linear programming fit your problem, use it.
If not, don't."***

He sat down. The field of Linear Programming was born.

For more historical readings, read "REMINISCENCES ABOUT THE ORIGINS OF LINEAR PROGRAMMING" by Dantzig himself!

The Toy Factory Example

Scenario: You build two products: **Widgets** (x_1) and **Gadgets** (x_2).

Profits:

- Widget: \$3 profit
- Gadget: \$4 profit

Constraints:

- **Metal:** Have 10kg. Widget uses 1, Gadget uses 2.
- **Wood:** Have 15kg. Widget uses 2, Gadget uses 1.

The LP Model:

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & 1x_1 + 2x_2 \leq 10 \\ & 2x_1 + 1x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The Canonical Form

Every LP can be written in Matrix Notation: $\max \mathbf{c}^T\mathbf{x}$ s.t. $\mathbf{A}\mathbf{x} \leq \mathbf{b}$.

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For our Factory:

$$\underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}^T}_{\mathbf{c}^T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}}$$

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- **x**: Decision Variables (The knobs we turn).
- **c**: Objective Coefficients (Profits/Costs).
- **A**: Constraint Matrix (Resource usage).
- **b**: Right-Hand Side (Capacities).

Pathologies: When things go wrong

Before we solve it, what if we *can't*?

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1. Infeasibility

No solution satisfies all constraints.

$$x \leq 2 \quad \text{AND} \quad x \geq 3$$

The feasible region is **Empty**.
Gurobi: `Model is infeasible.`

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The feasible region is **Empty**.
Gurobi: `Model is infeasible.`

2. Unboundedness

The region is open in the direction of improvement.

$$\max x \quad \text{s.t.} \quad x \geq 5$$

You can increase profit to ∞.
Gurobi: `Model is unbounded.`

Geometry: The Feasible Region

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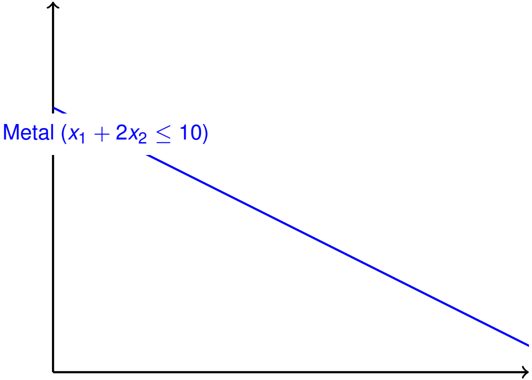

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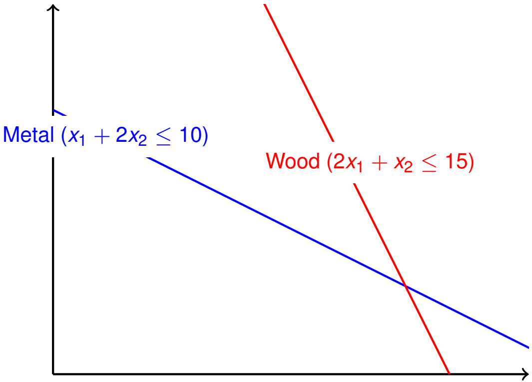

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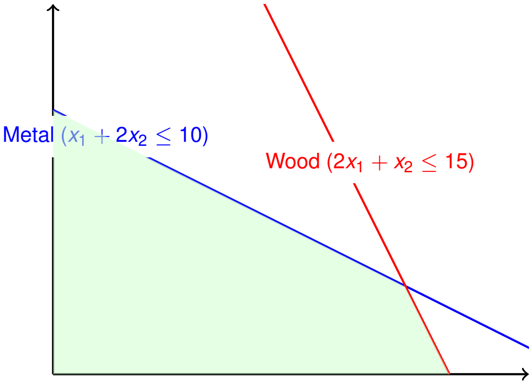

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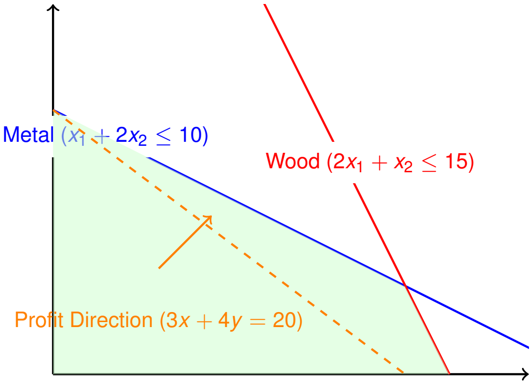

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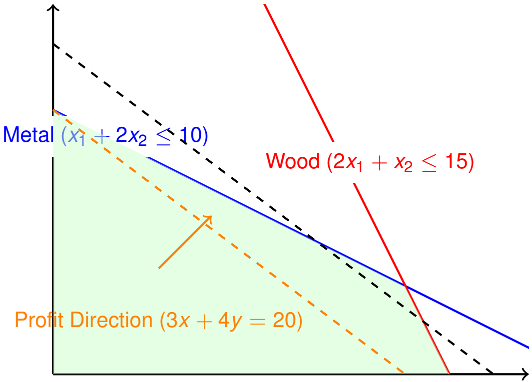

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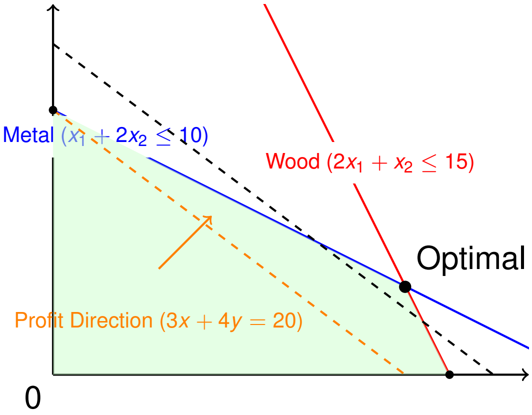
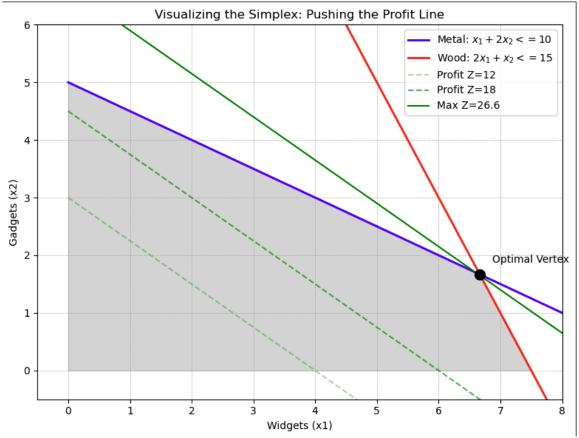
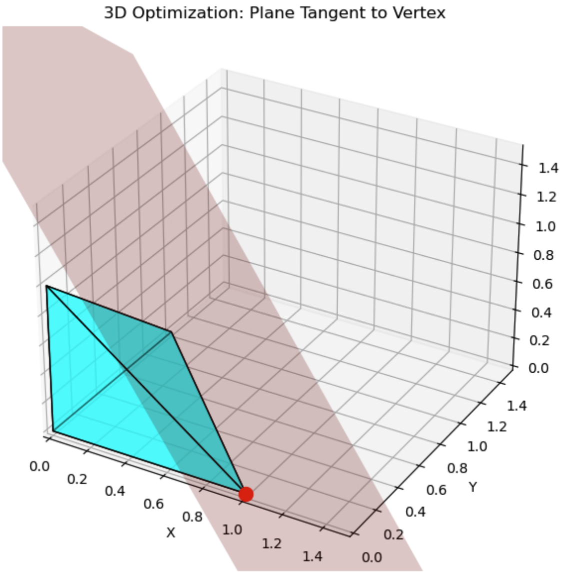


2D


Visualizing the Simplex: Pushing the Profit Line
Metal: $x_1 + 2x_2 <= 10$
Wood: $2x_1 + x_2 <= 15$
Profit Z=12
Profit Z=18
Max Z=26.6
Optimal Vertex
Gadgets (x2)
Widgets (x1)
0
1
2
3
4
5
6
7
8

3D


3D Optimization: Plane Tangent to Vertex
X
Y
0.0
0.2
0.4
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0.8
1.0
1.2
1.4

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Theorem

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1. Any point x in the polytope is a weighted average (convex combination) of the polytope's vertices $v_1, \ldots, v_k$: $x = \sum \alpha_i v_i$ with $\sum_i \alpha_i = 1, \alpha_i \geq 0$.

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4. An average cannot be larger than the maximum of its components.
5. Therefore, $f(x) \leq \max_i f(v_i)$. The max is at a corner!

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m.optimize()
print(x1.X, x2.X)
```

The Scenario

OptiMeal Inc. has a conflict:

- **Finance Team:** “Cut costs! Food is too expensive.”
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- **Objective:** Min Cost.
- **Constraints:** Calorie floor, Protein floor, Sugar ceiling, etc.

The Data (Nutrition & Costs)

Food	Cost ($)	Cal	Prot (g)	Carb (g)	Sugar (g)	Fiber (g)	Fat (g)
Chicken	1.80	128	24.0	0.0	0.0	0.0	2.7
Banana	0.30	105	1.3	27.0	14.0	3.1	0.4
Yogurt	0.90	104	5.9	7.9	7.9	0.0	5.5
Beans	1.10	120	8.0	21.0	1.0	7.0	0.5
Spinach	0.40	7	0.9	1.1	0.1	0.7	0.1
Almonds	0.70	160	6.0	6.0	1.0	3.0	14.0

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Requirements:

- Calories ≥ 2000
- Protein ≥ 100g
- Fiber ≥ 50g
- Sugar ≤ 50g
- Fat ≤ 120g
- Sodium ≤ 2300mg

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$$\begin{aligned} \text{s.t.} \quad \sum_j \text{Cal}_j \cdot x_j &\geq 2000 \\ \sum_j \text{Prot}_j \cdot x_j &\geq 100 \end{aligned}$$

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\text{s.t.} \quad \sum_j \text{Cal}_j \cdot x_j &\geq 2000 \\
\sum_j \text{Prot}_j \cdot x_j &\geq 100 \\
\sum_j \text{Sugar}_j \cdot x_j &\leq 50
\end{aligned}$$

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 & \displaystyle\sum_j \text{Prot}_j \cdot x_j & \geq & 100 \\
 & \displaystyle\sum_j \text{Sugar}_j \cdot x_j & \leq & 50 \\
 & \ldots & & \\
 & x_j & \geq & 0
\end{array}
$$

Implementation in Gurobi

```
params = {
    "Chicken" : { "price": 1.80, "protein": 24.0, "sugar": 0.0, "..." : "..." },
    "Banana"  : { "price": 0.30, "protein": 1.3,  "sugar": 14.0, "..." : "..." },
    "Yogurt"  : { "price": 0.90, "protein": 5.9,  "sugar": 7.9,  "..." : "..." },
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    ...
}
foods = list(params.keys())
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foods = list(params.keys())
# Variables: x[food]
x = m.addVars(foods, lb=0.0, name="servings")
```

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# Objective: Minimize Cost
obj_expr = 0
for food in foods:
    obj_expr += params[food]["price"] * x[i]
m.setObjective(obj_expr, GRB.MINIMIZE)
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# Constraints (Example: Protein & Sugar)
const_protein = m.addConstr(
                    gp.quicksum( params[fd]["protein"] * x[fd] for fd in foods) >= 100, "min_protein"
                )
```

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const_sugar = m.addConstr(gp.quicksum(params[food]["sugar"]  * x[i] for i in foods) <= 50, "max_sugar")
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params = {
    "Chicken" : { "price": 1.80, "protein": 24.0, "sugar": 0.0, "..." : "..." },
    "Banana"  : { "price": 0.30, "protein": 1.3,  "sugar": 14.0, "..." : "..." },
    "Yogurt"  : { "price": 0.90, "protein": 5.9,  "sugar": 7.9,  "..." : "..." },
    "Beans"   : { "price": 1.10, "protein": 8.0,  "sugar": 1.0,  "..." : "..." },
    ...
}
foods = list(params.keys())
# Variables: x[food]
x = m.addVars(foods, lb=0.0, name="servings")
# Objective: Minimize Cost
obj_expr = 0
for food in foods:
    obj_expr += params[food]["price"] * x[i]
m.setObjective(obj_expr, GRB.MINIMIZE)
# Constraints (Example: Protein & Sugar)
const_protein = m.addConstr(
                    gp.quicksum( params[fd]["protein"] * x[fd] for fd in foods) >= 100, "min_protein"
                )
const_sugar = m.addConstr(gp.quicksum(params[food]["sugar"]  * x[i] for i in foods) <= 50, "max_sugar")
# ... rest of the requirements ...
m.optimize()
```

Reading the Tea Leaves (Gurobi Output)

When you run `m.optimize()`, Gurobi populates attributes on the objects.

Model Attributes:

- `m.Status`: Did it work?
  (2=Opt, 3=Infeas, 5=Unbdd)

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- `m.ObjVal`: The total profit/cost (Z).

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  ($x_1 = 6.66$).

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- `var.RC`: **Reduced Cost**. How much the objective coefficient must improve before this variable becomes non-zero (More next week).

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- `constr.Slack`: Difference between LHS and RHS.

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- `constr.Slack`: Difference between LHS and RHS.
- `constr.Pi` (π): **Shadow Price**. *"If I had 1 more unit of Metal, how much more profit would I make?". More on this next week!*

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Constraint Attributes:

- `constr.Slack`: Difference between LHS and RHS.
- `constr.Pi` (π): **Shadow Price**. *"If I had 1 more unit of Metal, how much more profit would I make?". More on this next week!*

Warning

Attributes like `.X` and `.Pi` are only available if `m.Status == 2` (Optimal). Always check status first!

Infeasibility Diagnosis

```python
import gurobipy as gp
import gurobipy

m = gp.Model("Infeasible")
x = m.addVar(name="x")
m.setObjective(-1*x, gp.GRB.MAXIMIZE)
m.addConstr(x>=3)
m.addConstr(x<=2)
m.optimize()
print("Optimize status:", m.Status)
```

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x = m.addVar(name="x")
m.setObjective(-1*x, gp.GRB.MAXIMIZE)
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```

```
Gurobi Optimizer version 12.0.3 build v12.0.3rc0 (mac64[arm]
- Darwin 23.1.0 23B2073)

CPU model: Apple M3 Max
Thread count: 14 physical cores, 14 logical processors, using up to 14 threa

Optimize a model with 2 rows, 1 columns and 2 nonzeros
Model fingerprint: 0xf5b06d2b
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [0e+00, 0e+00]
  RHS range        [2e+00, 3e+00]
Presolve time: 0.00s

Solved in 0 iterations and 0.00 seconds (0.00 work units)
Infeasible model
Optimize status: 3
```

What about Larger Models?

```
import gurobipy as gp

m = gp.Model("TrickyInfeasible")

# Variables
x = m.addVar(lb=-0, ub=8, name="x")
y = m.addVar(lb=-0, ub=8, name="y")
```

What about Larger Models?

```python
import gurobipy as gp

m = gp.Model("TrickyInfeasible")

# Variables
x = m.addVar(lb=-0, ub=8, name="x")
y = m.addVar(lb=-0, ub=8, name="y")
# Arbitrary bounded objective
m.setObjective(x + y, gp.GRB.MINIMIZE)
```

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x = m.addVar(lb=-0, ub=8, name="x")
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# Arbitrary bounded objective
m.setObjective(x + y, gp.GRB.MINIMIZE)
#Constraints
m.addConstr(2*x + y <= 4, name="c1_budget1")
m.addConstr(x + 2*y <= 4, name="c2_budget2")
m.addConstr(x + y >= 5,   name="c3_demand")
m.addConstr(x <= 8,       name="c4_x_cap")
m.addConstr(y <= 8,       name="c5_y_cap")
```

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m.addConstr(x <= 8,       name="c4_x_cap")
m.addConstr(y <= 8,       name="c5_y_cap")
m.optimize()
print("Optimize status:", m.Status)
```

```
Optimize a model with 5 rows, 2 columns and 8 nonzeros
Model fingerprint: 0x00fc1d77
Coefficient statistics:
  Matrix range     [1e+00, 2e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [8e+00, 8e+00]
  RHS range        [4e+00, 8e+00]
Presolve removed 2 rows and 0 columns
Presolve time: 0.01s

Solved in 0 iterations and 0.01 seconds (0.00 work units)
Infeasible model
Optimize status: 3
```

Irreducible Infeasible Subsystem (IIS)

What is an IIS?

- When a model is **infeasible**, the full set of constraints cannot all be satisfied simultaneously.

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- An **IIS** is a *minimal subset of constraints and bounds* that is still infeasible.

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- When a model is **infeasible**, the full set of constraints cannot all be satisfied simultaneously.
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Irreducible Infeasible Subsystem (IIS)

What is an IIS?

- When a model is **infeasible**, the full set of constraints cannot all be satisfied simultaneously.
- An **IIS** is a *minimal subset of constraints and bounds* that is still infeasible.
- "Minimal" = removing *any* constraint from that subset makes it feasible again.
- IISs help pinpoint the true source of infeasibility in large models.

Good News

Gurobi can compute an IIS for you automatically!

Computing an IIS in Gurobi

If the model is infeasible, we can ask Gurobi to identify the conflicting constraints.

Computing an IIS in Gurobi

If the model is infeasible, we can ask Gurobi to identify the conflicting constraints.

```
if m.Status == GRB.INFEASIBLE:
    print("\nModel is infeasible; computing IIS...")
    m.computeIIS()

    print("Constraints in the IIS:")
    for c in m.getConstrs():
        if c.IISConstr:   # True if part of the IIS
            print(f"  {c.ConstrName}")
```

Example IIS Output

```
Model is infeasible; computing IIS...

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    0.0000000e+00   2.500000e+00   0.000000e+00      0s

IIS computed: 3 constraints and 0 bounds
IIS runtime: 0.00 seconds (0.00 work units)
Constraints in the IIS:
  c1_budget1
  c2_budget2
  c3_demand
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Example IIS Output

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```

- Remember, these constraints correspond to $2x + y \leq 4$, $x + 2y \leq 4$, and $x + y \geq 5$. Adding the first 2 inequalities contradicts the third.

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- These are the **minimal conflicting constraints**.
- Removing any one of them would make the model feasible.
- Great for isolating modeling mistakes in large LPs/MIPs.

Unbounded LPs and Infinite Directions

Unbounded LP = The objective can grow without limit while staying feasible.

Unbounded LPs and Infinite Directions

Unbounded LP = The objective can grow without limit while staying feasible.
Gurobi not only detects unboundedness, it returns an *unbounded ray*.

- An **unbounded ray** is a vector d such that:

$$x + \lambda d \text{ is feasible for all } \lambda > 0$$

  and the objective coefficient $c^T d > 0$ (for maximization).

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  ```
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  ```

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Interpretation

The unbounded ray shows *how* the LP escapes to infinity.

Example of an Unbounded LP?

Example (Maximization):

$$\max \; x + y$$

$$\text{s.t. } x - y \geq 1$$

$$x, y \geq 0$$

Example of an Unbounded LP?

Example (Maximization):

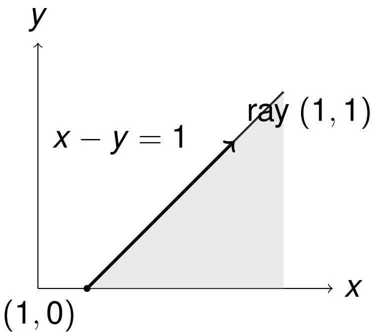
$$\max \; x + y$$

$$\text{s.t. } x - y \geq 1$$

$$x, y \geq 0$$

- Feasible region goes to ∞.
- Objective increases without bound.
- No vertex optimum exists.

Geometry of the Unbounded Ray

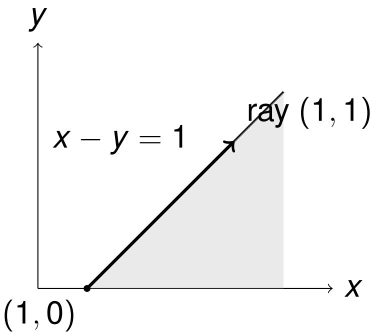


Feasible region:

$$x - y \geq 1, \quad x \geq 0, \quad y \geq 0.$$

- From the feasible point $(1, 0)$ we can move along $(x, y) = (1, 0) + \lambda(1, 1) = (1 + \lambda, \lambda), \ \lambda \geq 0$.

Geometry of the Unbounded Ray



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$$1 + 2\lambda \to \infty.$$

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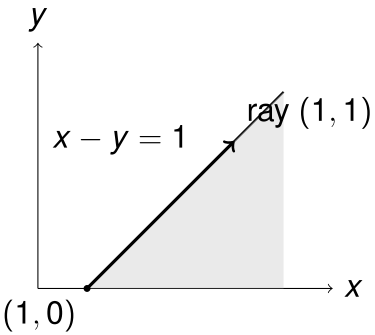

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- The objective $x + y$ grows without bound:
$$1 + 2\lambda \to \infty.$$
- Gurobi's `UnbdRay` returns this direction.

Gurobi Example: Unbounded Model + Ray

```python
import gurobipy as gp
from gurobipy import GRB

m = gp.Model("Unbounded")
x = m.addVar(lb=0, name="x")
y = m.addVar(lb=0, name="y")
m.setObjective(x + y, GRB.MAXIMIZE)
m.addConstr(x - y >= 1, name="c1_skew")
# KEY: ask Gurobi to compute ray info
m.setParam(GRB.Param.InfUnbdInfo, 1)
m.optimize()
```

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m.addConstr(x - y >= 1, name="c1_skew")
# KEY: ask Gurobi to compute ray info
m.setParam(GRB.Param.InfUnbdInfo, 1)
m.optimize()

print("Status:", m.Status)
if m.Status == GRB.UNBOUNDED:
    print("\nUnbounded Ray:")
    for v in m.getVars():
        print(f"{v.VarName}: {v.UnbdRay}")
```

Gurobi Example: Unbounded Model + Ray

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from gurobipy import GRB

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m.optimize()

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if m.Status == GRB.UNBOUNDED:
    print("\nUnbounded Ray:")
    for v in m.getVars():
        print(f"{v.VarName}: {v.UnbdRay}")
```

```
Status: 5

Unbounded Ray:
x: 1.0
y: 1.0
```

Gurobi Example: Unbounded Model + Ray

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import gurobipy as gp
from gurobipy import GRB

m = gp.Model("Unbounded")
x = m.addVar(lb=0, name="x")
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# KEY: ask Gurobi to compute ray info
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m.optimize()

print("Status:", m.Status)
if m.Status == GRB.UNBOUNDED:
    print("\nUnbounded Ray:")
    for v in m.getVars():
        print(f"{v.VarName}: {v.UnbdRay}")
```

```
Status: 5

Unbounded Ray:
x: 1.0
y: 1.0
```

- The ray $(1, 1)$ means both x and y can increase indefinitely.
- The constraint $x - y \geq 1$ stays satisfied for all $(x, y) = (1, 0) + \lambda(1, 1)$.
- Objective grows as $x + y \to +\infty$.

TODOs after Lecture.

- **Install Gurobi:** Get your academic license working.
- Code and Solve **The Diet Problem** in HW1.
- Use **Tools** like `m.computeIIS()` and `var.UnbdRay` to find the conflict in toy infeasible models and unbounded models.