CS498: Algorithmic Engineering

Lecture 2: Simplex & Duality

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Outline

How do Solvers actually work?

Last lecture, we defined the LP:

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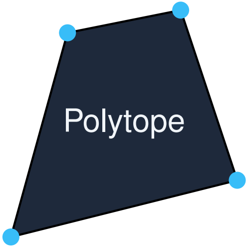

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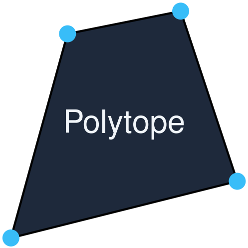

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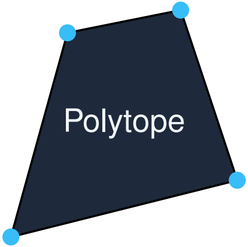

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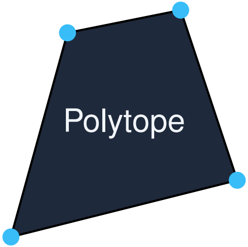
  Problem: A hypercube in n dimensions has 2^n vertices. Too slow.



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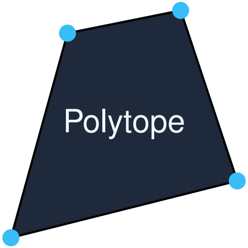

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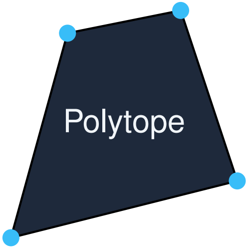

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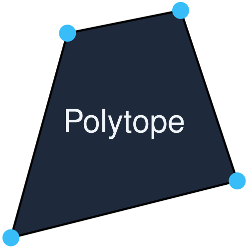

Key Insight:

- Vertices are connected by **edges**.
- We can "walk" from vertex to vertex improving our objective.

The Simplex Intuition (Hill Climbing)

Algorithm (Dantzig, 1947):

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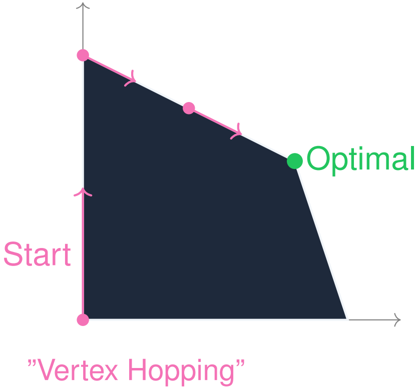

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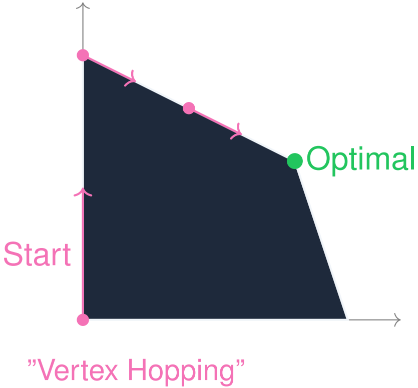

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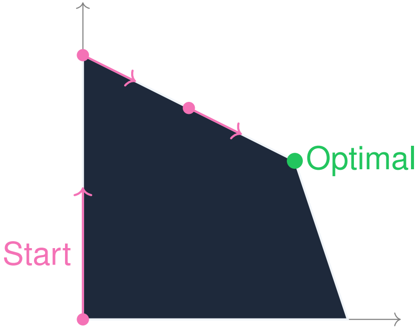
Algorithm (Dantzig, 1947):

1. **Start** at any vertex (usually Origin).
2. **Look** along edges connected to current vertex.
3. **Is a neighbor better?**
   - **Yes:** Move there (Pivot). Go to 2.
   - **No:** You are done. (Local max = Global max).



"Vertex Hopping"

Geometric View: A 2D Example

Example: $\max 3x_1 + 4x_2$ subject to: $x_1 + 2x_2 \leq 10, 2x_1 + x_2 \leq 15, x_1, x_2 \geq 0$.

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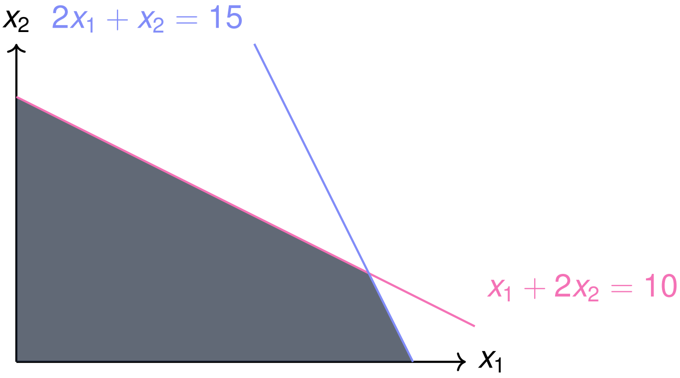

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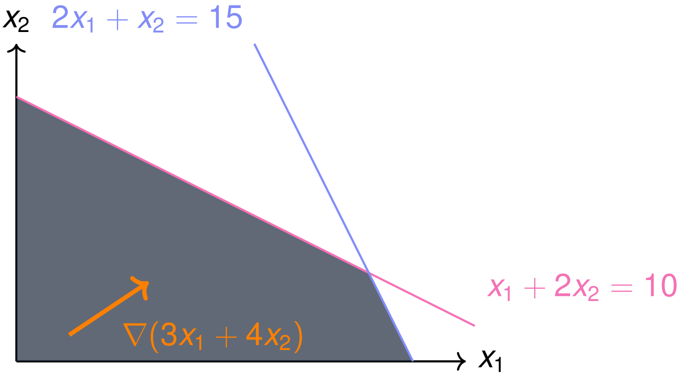

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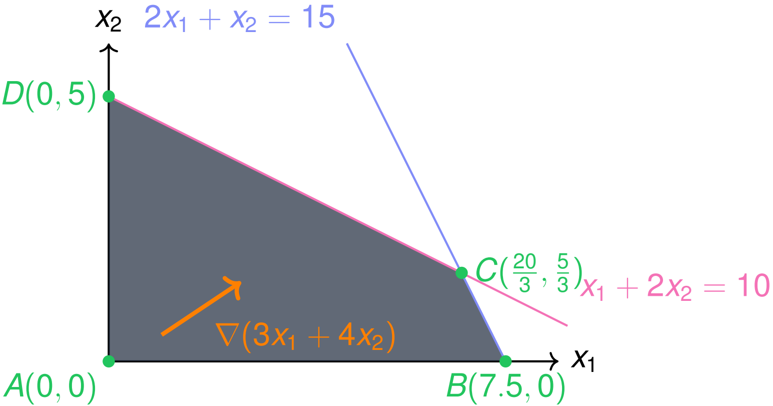

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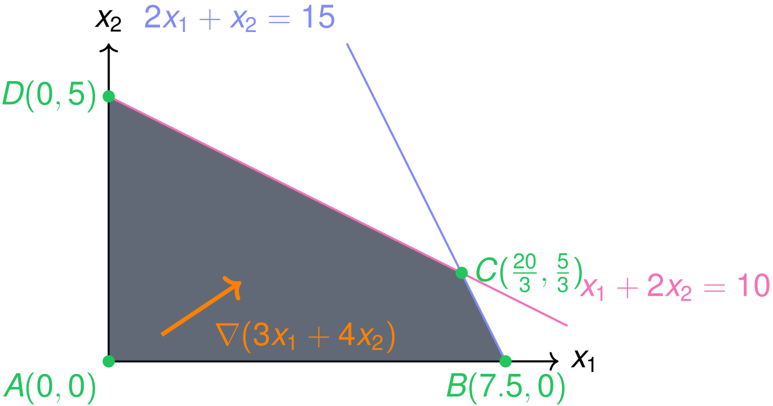
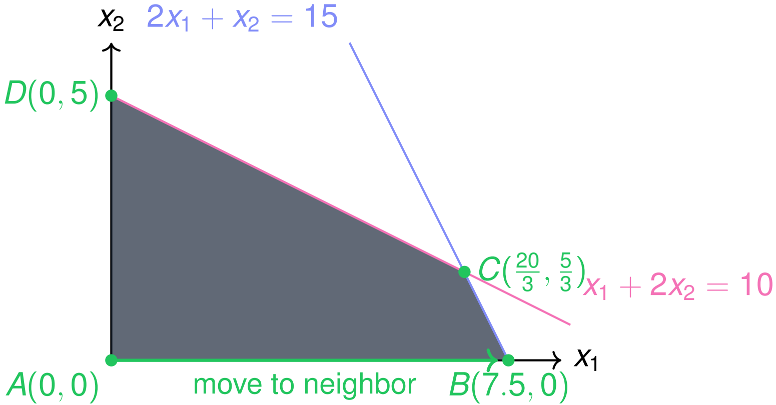


Simplex: pick a basic feasible solution (a vertex), $A(0, 0)$ with $z = 3x_1 + 4x_2 = 0$.

Geometric View: Simplex Walk (Step 1)

Start at $A = (0, 0)$, $z = 0$. Check neighboring vertices on the polytope.



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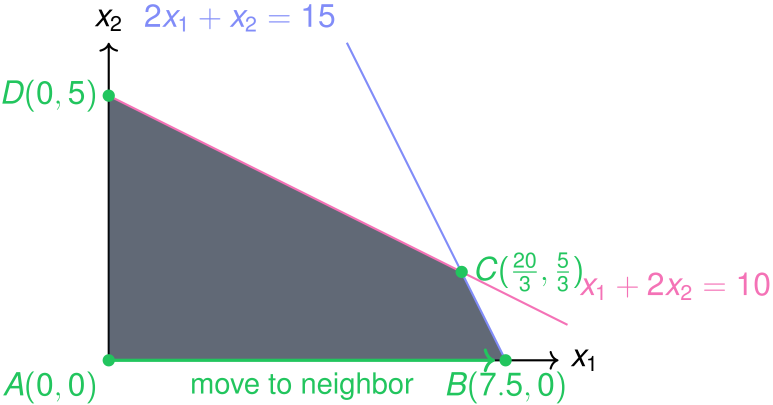

At $B = (7.5, 0)$: $z = 3 \cdot 7.5 + 4 \cdot 0 = 22.5 > 0$, so simplex pivots from A to B.

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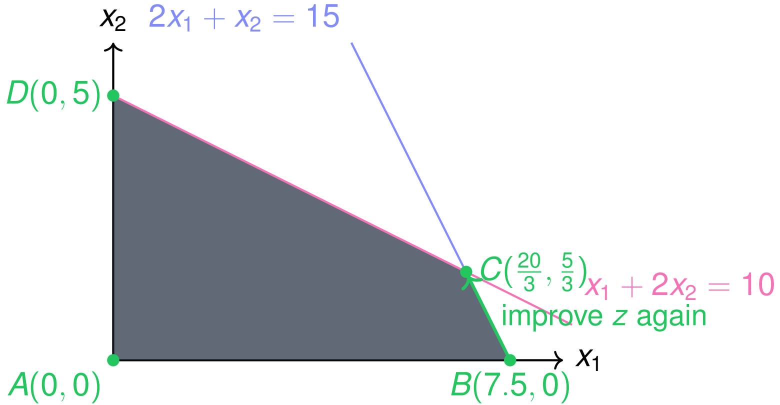

At $C = \left(\frac{20}{3}, \frac{5}{3}\right), z = 3x_1 + 4x_2 = 3 \cdot \frac{20}{3} + 4 \cdot \frac{5}{3} = \frac{80}{3} \approx 26.7$

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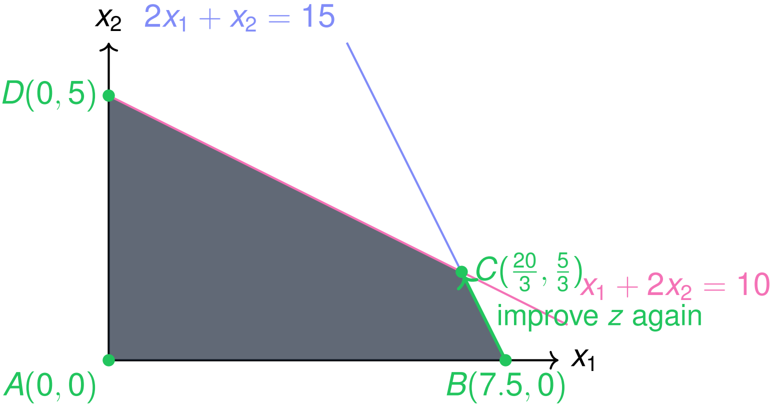

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No neighbor improves z further $\Rightarrow$ simplex stops: C is optimal.

Vertices and Basic Feasible Solutions

Setup: Consider the LP in two variables:

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Key idea: In 2D, a vertex is the intersection of **2 boundaries**.

From Geometry to Algebra

Boundaries = constraints tight as equalities.

In our example, boundaries include:

$$x_1 = 0, \quad x_2 = 0, \quad x_1 + 2x_2 = 10, \quad 2x_1 + x_2 = 15.$$

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In general:

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In general:

- In an LP with *m* variables, a vertex comes from *m* **tight constraints**.
- Solving those *m* equalities gives a single point.

Examples of Vertices as Intersections

Vertex $A = (0, 0)$

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Basic Feasible Solutions (Algebraic View)

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A point is a **basic feasible solution (BFS)** if:

- It is **feasible** (satisfies all constraints).
- It is a **vertex** of the feasible region:
  - in m dimensions: it lies at the intersection of m **tight constraints** (including $x_j \geq 0$), and those m equalities have a unique solution.

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Key facts:

- Every vertex $\iff$ a BFS.
- Simplex method moves from one BFS (vertex) to another, improving the objective each time.

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Goal: choose the edge that **increases** z.

Step 1: Which Edge Improves the Objective?

Edge 1: keep $x_1 = 0$, relax $x_1 + 2x_2 = 10$ to $x_1 + 2x_2 \leq 10 \implies x_2 \leq 5$.

On this edge:

$$x_1 = 0, \quad x_2 = 5 - t, \quad t \geq 0 \quad \text{(moving down from } D\text{)}.$$

Objective:

$$z(t) = 3 \cdot 0 + 4(5 - t) = 20 - 4t.$$

Slope: z **decreases** as t increases.

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Edge 2: keep $x_1 + 2x_2 = 10$, relax $x_1 = 0$ to $x_1 \geq 0$.
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$$x_1 = t, \quad x_2 = \frac{10 - t}{2}, \quad t \geq 0 \quad \text{(moving right from } D\text{)}.$$

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So we pivot along **Edge 2**. (Entering variable: x_1.)

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Constraint $x_2 \geq 0$: $\frac{10-t}{2} = 0 \;\Rightarrow\; t = 10.$

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Constraint $x_2 \geq 0$: $\frac{10-t}{2} = 0 \;\Rightarrow\; t = 10$.

Constraint $x_1 \geq 0$ is fine for all $t \geq 0$.

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Constraint $x_2 \geq 0$: $\frac{10-t}{2} = 0 \;\Rightarrow\; t = 10.$

Constraint $x_1 \geq 0$ is fine for all $t \geq 0$.

$$\text{Smallest nonnegative } t \text{ is } \frac{10}{3}\text{. So the next constraint to become tight is:}$$

$$2x_1 + x_2 = 15.$$

Step 3: New Vertex = Solve a 2 × 2 System

At the new vertex, the tight constraints are:

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New BFS (after the pivot):

$$C = \left(\frac{20}{3}, \frac{5}{3}\right).$$

A pivot is literally: change one equation, solve a tiny linear system.

Repeat the Same Three Steps at the New Vertex

At C there are again two tight constraints (two equations). To pivot again, we repeat the same pattern:

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1. **Choose an edge:** Relax one tight constraint, keep the other. Parametrize the edge, compute $z(t)$, pick the edge with increasing z.
2. **Ratio test:** Plug the parametrization into all constraints, solve for t, and find which constraint hits equality first.
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In m dimensions, it's the same idea: solve m equations, relax one, ratio test, solve a new $m \times m$ system.

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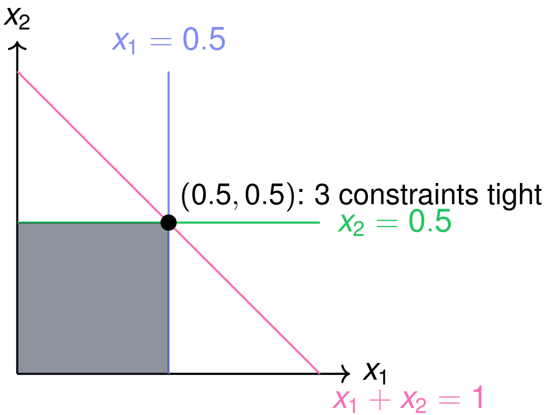
This situation is called **degeneracy**: the vertex is the unique solution of more equations than needed.

Degeneracy: Visualizing the Point $(0.5, 0.5)$

Constraints:

$$x_1 + x_2 \leq 1, \quad x_1 \leq 0.5, \quad x_2 \leq 0.5, \quad x_1 \geq 0, \quad x_2 \geq 0$$



At $(0.5, 0.5)$, **all three** constraints are tight, but it is still just *one* vertex $\Rightarrow$ **degeneracy**.

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If this happens repeatedly, simplex can "stall" or even cycle (cycling is fixed in practice by Bland's rule and similar tie-breakers).

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- In practice: much fewer iterations needed
- Typical: $O(m)$ to $O(m \log n)$ iterations

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Same exact ideas, but more efficient.

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Our focus is on the modelling. Gurobi handles the actual solver implementation.

Motivation: Finding Upper Bounds

Example LP:

$$\max 4x_1 + x_2 + 3x_3 \quad \text{s.t.} \quad x_1 + 4x_2 \leq 1, \quad 3x_1 - x_2 + x_3 \leq 3, \quad x_1, x_2, x_3 \geq 0$$

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So $Z \le 11$. We've bounded the optimum: $9 \le Z \le 11$.

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Then: $4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + y_2x_3 \leq y_1 + 3y_2$

Goal: Minimize $y_1 + 3y_2$ subject to those constraints on y!

The Dual Problem Emerges

We naturally arrived at:

Primal (P)

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & x_1 + 4x_2 \leq 1 \\ & 3x_1 - x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Dual (D)

$$\begin{aligned} \min \quad & y_1 + 3y_2 \\ \text{s.t.} \quad & y_1 + 3y_2 \geq 4 \\ & 4y_1 - y_2 \geq 1 \\ & y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

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Key Insight:

- Every feasible y gives an upper bound on the primal optimum
- The dual finds the *best* (tightest) upper bound
- This is **duality theory**!

General Duality: Matrix Form

Primal (P)	Dual (D)
$\max c^T x$	$\min b^T y$
$Ax \leq b$	$A^T y \geq c$
$x \geq 0$	$y \geq 0$

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- Primal $\max$ $\leftrightarrow$ Dual $\min$
- Constraint matrix transposes: $A \to A^T$

Duality Rules: The Full Picture

Primal (max $c^\top x$)	**Dual (min $b^\top y$)**
$\sum_j a_{ij} x_j \leq b_i$	$y_i \geq 0$
$\sum_j a_{ij} x_j \geq b_i$	$y_i \leq 0$
$\sum_j a_{ij} x_j = b_i$	y_i free
$x_j \geq 0$	$\sum_i y_i a_{ij} \geq c_j$
$x_j \leq 0$	$\sum_i y_i a_{ij} \leq c_j$
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x_j free	$\sum_i y_i a_{ij} = c_j$

Key symmetry: the dual of the dual is your original primal.

Example 1: Building the Dual Step by Step

Primal:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Example 1: Building the Dual Step by Step

Primal:

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We now construct the dual using the duality rules.

Example 1: Step 1 — Dual Variables

Primal constraint types $\Rightarrow$ dual variable signs

Primal (for reference)

$$
\begin{aligned}
\max\ & 5x_1 + 3x_2 \\
\text{s.t. } & 2x_1 + x_2 \le 8 \\
& x_1 + 3x_2 \le 9 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \le b_i$	$y_i \ge 0$
$a_i^\top x \ge b_i$	$y_i \le 0$
$a_i^\top x = b_i$	y_i free

Example 1: Step 1 — Dual Variables

Primal constraint types $\Rightarrow$ dual variable signs

$$2x_1 + x_2 \le 8 \Rightarrow y_1 \ge 0,$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t.}\ & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \le b_i$	$y_i \ge 0$
$a_i^\top x \ge b_i$	$y_i \le 0$
$a_i^\top x = b_i$	y_i free

Example 1: Step 1 — Dual Variables

Primal constraint types $\Rightarrow$ dual variable signs

$$2x_1 + x_2 \le 8 \Rightarrow y_1 \ge 0,$$

$$x_1 + 3x_2 \le 9 \Rightarrow y_2 \ge 0.$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix},\ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \le b_i$	$y_i \ge 0$
$a_i^\top x \ge b_i$	$y_i \le 0$
$a_i^\top x = b_i$	y_i free

Example 1: Step 2 — Dual Objective

Objective direction:

$$\max \implies \min .$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \; b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \; c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \leq b_i$	$y_i \geq 0$
$a_i^\top x \geq b_i$	$y_i \leq 0$
$a_i^\top x = b_i$	y_i free

Example 1: Step 2 — Dual Objective

Objective direction:

$$\max \implies \min .$$

Dual objective uses the RHS values:

$$b = (8,\ 9).$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix},\ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \le b_i$	$y_i \ge 0$
$a_i^\top x \ge b_i$	$y_i \le 0$
$a_i^\top x = b_i$	y_i free

Example 1: Step 2 — Dual Objective

Objective direction:

$$\max \implies \min .$$

Dual objective uses the RHS values:

$$b = (8,\ 9).$$

So the dual objective is

$$\min\ 8y_1 + 9y_2.$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix},\ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \le b_i$	$y_i \ge 0$
$a_i^\top x \ge b_i$	$y_i \le 0$
$a_i^\top x = b_i$	y_i free

Example 1: Step 3 — Dual Constraints

Primal variable signs $\Rightarrow$ dual constraints

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t.}\ & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix},\ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \ge 0$	$(A^\top y)_j \ge c_j$
$x_j \le 0$	$(A^\top y)_j \le c_j$
x_j free	$(A^\top y)_j = c_j$

Example 1: Step 3 — Dual Constraints

Primal variable signs $\Rightarrow$ dual constraints

$x_1 \geq 0$:

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix},\ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 1: Step 3 — Dual Constraints

Primal variable signs ⇒ dual constraints

$x_1 \geq 0$:

$$2y_1 + y_2 \geq 5$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix},\ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix},\ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 1: Step 3 — Dual Constraints

Primal variable signs $\Rightarrow$ dual constraints

$x_1 \geq 0$:

$$2y_1 + y_2 \geq 5$$

$x_2 \geq 0$:

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \ b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \ c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 1: Step 3 — Dual Constraints

Primal variable signs ⇒ dual constraints

$x_1 \geq 0$:

$$2y_1 + y_2 \geq 5$$

$x_2 \geq 0$:

$$y_1 + 3y_2 \geq 3$$

Primal (for reference)

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t.}\ & 2x_1 + x_2 \leq 8 \\ & x_1 + 3x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Matrix summary

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \quad c = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 1: Final Dual

$$\boxed{\begin{aligned} \min\ & 8y_1 + 9y_2 \\ \text{s.t. } & 2y_1 + y_2 \geq 5 \\ & y_1 + 3y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}}$$

Example 1: Final Dual

$$\boxed{\begin{aligned} \min\ & 8y_1 + 9y_2 \\ \text{s.t. } & 2y_1 + y_2 \geq 5 \\ & y_1 + 3y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}}$$

Every coefficient comes directly from the primal via the duality rules.

Example 2: Mixed Constraints — Dual Variables

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ \ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \leq b_i$	$y_i \geq 0$
$a_i^\top x \geq b_i$	$y_i \leq 0$
$a_i^\top x = b_i$	y_i free

Example 2: Mixed Constraints — Dual Variables

Constraint types $\Rightarrow$ dual variable signs

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t.}\ & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ \ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \leq b_i$	$y_i \geq 0$
$a_i^\top x \geq b_i$	$y_i \leq 0$
$a_i^\top x = b_i$	y_i free

Example 2: Mixed Constraints — Dual Variables

Constraint types $\Rightarrow$ dual variable signs

$$x_1 + x_2 + x_3 = 10 \Rightarrow y_1 \text{ free}$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \leq b_i$	$y_i \geq 0$
$a_i^\top x \geq b_i$	$y_i \leq 0$
$a_i^\top x = b_i$	y_i free

Example 2: Mixed Constraints — Dual Variables

Constraint types $\Rightarrow$ dual variable signs

$$x_1 + x_2 + x_3 = 10 \Rightarrow y_1 \text{ free}$$

$$2x_1 + x_2 \geq 5 \Rightarrow y_2 \leq 0$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ \ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \leq b_i$	$y_i \geq 0$
$a_i^\top x \geq b_i$	$y_i \leq 0$
$a_i^\top x = b_i$	y_i free

Example 2: Mixed Constraints — Dual Variables

Constraint types $\Rightarrow$ dual variable signs

$$x_1 + x_2 + x_3 = 10 \Rightarrow y_1 \text{ free}$$

$$2x_1 + x_2 \geq 5 \Rightarrow y_2 \leq 0$$

Dual objective:

$$\min \; 10y_1 + 5y_2.$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0, \;\; x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, \; c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$a_i^\top x \leq b_i$	$y_i \geq 0$
$a_i^\top x \geq b_i$	$y_i \leq 0$
$a_i^\top x = b_i$	y_i free

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

$x_1 \geq 0$:

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0, \ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, \ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

$x_1 \geq 0$:

$$1 \cdot y_1 + 2 \cdot y_2 \geq 4$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

$x_1 \geq 0$:

$$1 \cdot y_1 + 2 \cdot y_2 \geq 4$$

$x_2 \geq 0$:

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

$x_1 \geq 0$:

$$1 \cdot y_1 + 2 \cdot y_2 \geq 4$$

$x_2 \geq 0$:

$$1 \cdot y_1 + 1 \cdot y_2 \geq 2$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0, \ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix}, \ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

$x_1 \geq 0$:

$$1 \cdot y_1 + 2 \cdot y_2 \geq 4$$

$x_2 \geq 0$:

$$1 \cdot y_1 + 1 \cdot y_2 \geq 2$$

x_3 free:

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Mixed Constraints — Dual Constraints

Variable types $\Rightarrow$ dual constraints

$x_1 \geq 0$:

$$1 \cdot y_1 + 2 \cdot y_2 \geq 4$$

$x_2 \geq 0$:

$$1 \cdot y_1 + 1 \cdot y_2 \geq 2$$

x_3 free:

$$1 \cdot y_1 + 0 \cdot y_2 = 1$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Example 2: Final Dual

Final dual:

$$\boxed{\begin{aligned} &\min 10y_1 + 5y_2 \\ &\text{s.t } y_1 + 2y_2 \geq 4, \\ &\quad y_1 + y_2 \geq 2, \\ &\quad y_1 = 1, \\ &\quad y_2 \leq 0 \end{aligned}}$$

Primal:

$$\begin{aligned} \max\ & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 = 10 \\ & 2x_1 + x_2 \geq 5 \\ & x_1, x_2 \geq 0,\ x_3 \text{ free} \end{aligned}$$

Primal summary

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix},\ b = \begin{pmatrix} 10 \\ 5 \end{pmatrix},\ c = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

Duality rules

Primal	Dual
$\max c^\top x$	$\min b^\top y$
$x_j \geq 0$	$(A^\top y)_j \geq c_j$
$x_j \leq 0$	$(A^\top y)_j \leq c_j$
x_j free	$(A^\top y)_j = c_j$

Theorems of Duality

1. Weak Duality Theorem

For any feasible primal x and any feasible dual y:

$$c^T x \leq b^T y$$

Primal objective $\leq$ *Dual objective*

Proof: If $Ax \leq b$ and $A^T y \geq c$ with $x, y \geq 0$:

$$c^T x \leq (A^T y)^T x = y^T (Ax) \leq y^T b = b^T y \quad \square$$

Theorems of Duality

2. Strong Duality Theorem

If the Primal has an optimal solution x^*, then the Dual has an optimal solution y^*, and:

$$c^T x^* = b^T y^*$$

At optimality, the objectives are ***equal****—no gap!*

Theorems of Duality

2. Strong Duality Theorem

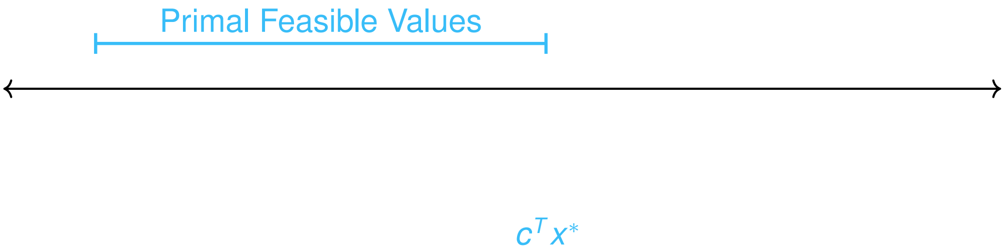
If the Primal has an optimal solution x^*, then the Dual has an optimal solution y^*, and:

$$c^T x^* = b^T y^*$$

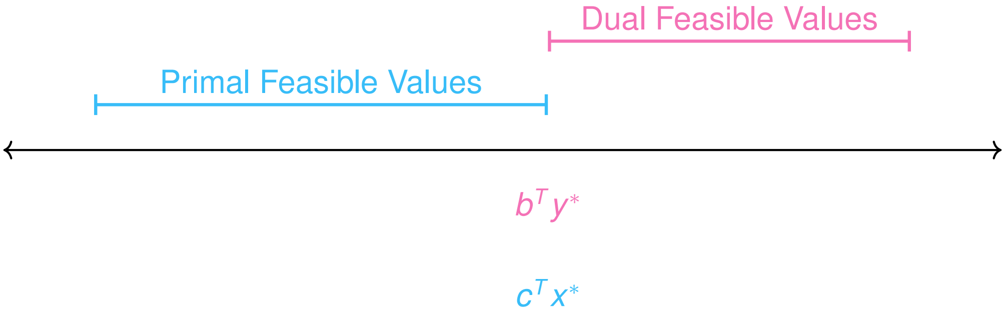
*At optimality, the objectives are **equal**—no gap!*

Note: Strong duality proof requires more machinery (Farkas' lemma), but the result is powerful.

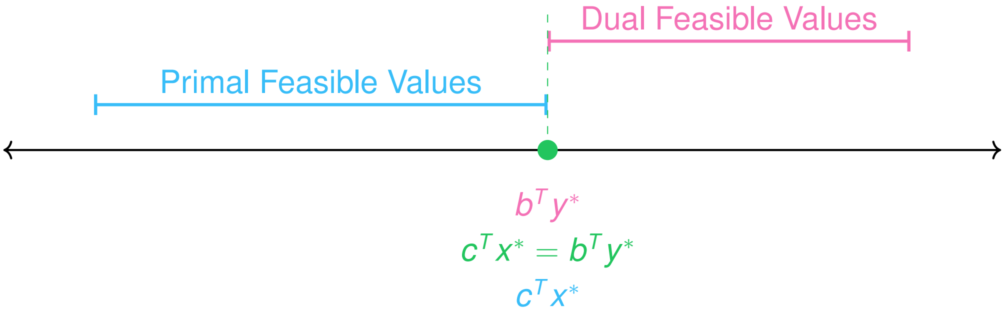
Visualizing Weak & Strong Duality



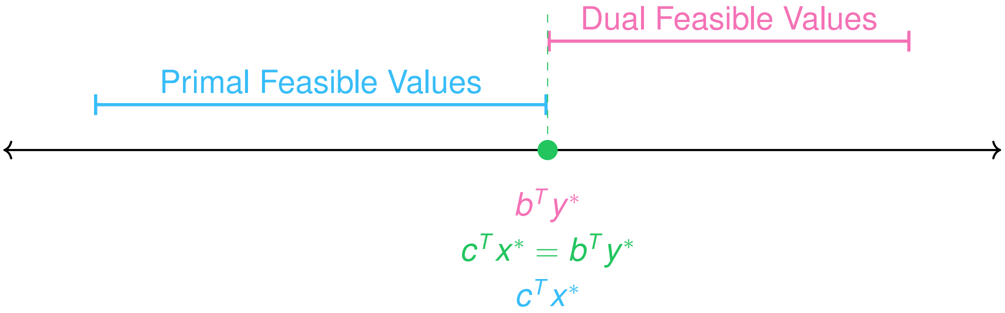
Visualizing Weak & Strong Duality



Visualizing Weak & Strong Duality



Visualizing Weak & Strong Duality



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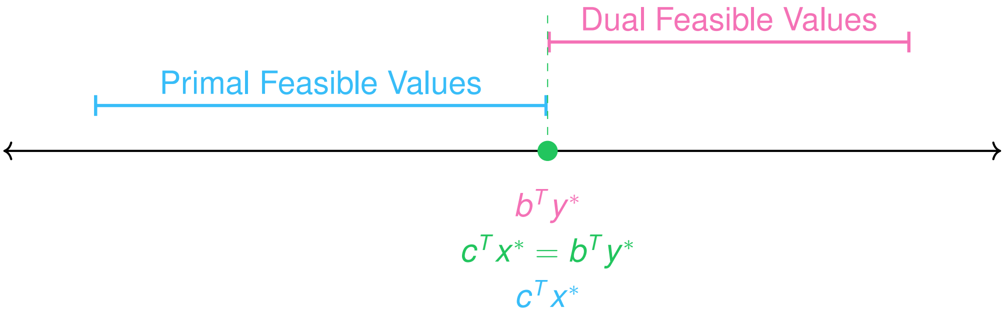

Key Insight:

- Any primal feasible $\leq$ any dual feasible (weak duality)
- At optimum, they meet exactly (strong duality)

Accessing Duals in Gurobi

We can use Gurobi to perform sensitivity analysis automatically.

```python
# ... (Model definition) ...
m.optimize()

print("Optimal Primal (Production):")
for v in m.getVars():
    print(f"{v.VarName}: {v.X}")

print("\nOptimal Dual:")
for c in m.getConstrs():
    # .Pi is the attribute for the Dual Variable (Price)
    print(f"{c.ConstrName}: {c.Pi}")
```

Example: Solving the Primal in Gurobi

Primal (Example 1):

$$\begin{aligned} \max\ & 5x_1 + 3x_2 \\ \text{s.t. } & 2x_1 + x_2 \le 8 \\ & x_1 + 3x_2 \le 9 \\ & x_1, x_2 \ge 0 \end{aligned}$$

```
m = gp.Model()
x1 = m.addVar(lb=0, name="x1")
x2 = m.addVar(lb=0, name="x2")
c1 = m.addConstr(2*x1 + x2 <= 8, name="c1")
c2 = m.addConstr(x1 + 3*x2 <= 9, name="c2")

m.setObjective(5*x1 + 3*x2, gp.GRB.MAXIMIZE)
m.optimize()

print("Optimal primal value:", m.ObjVal)
```

Example: Dual Values and Strong Duality

Dual of Example 1:

$$\begin{aligned} \min\ & 8y_1 + 9y_2 \\ \text{s.t. } & 2y_1 + y_2 \geq 5 \\ & y_1 + 3y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

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Gurobi gives the dual values as `constraint.Pi`:

```
print("Dual values (shadow prices):")
print("y1 =", c1.Pi)
print("y2 =", c2.Pi)

dual_obj = 8*c1.Pi + 9*c2.Pi
print("Dual objective:", dual_obj)
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Strong Duality Check

If you run the code, Gurobi returns:

$$x^* = (3,\ 2)$$
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Primal optimal = Dual optimal.
Strong duality verified!

Summary: What We Learned

The Simplex Algorithm:

- Geometrically: walks from vertex to vertex along edges
- Algebraically: Basic Feasible Solutions (BFS) via pivoting
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Duality Theory:

- Every LP has a dual that provides upper bounds
- Weak duality: primal $\leq$ dual always
- Strong duality: they meet at optimum (no gap!)
- Conversion rules for mixed constraint types