CS498: Algorithmic Engineering

Lecture 17: Introduction to SAT and Z3

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Outline

Where We Are in the Course

Part I (Lec. 1–9):

- LP, IP, B&B.
- Row generation, lazy cuts.
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Today: Part III begins. A new way of thinking about problems.

The Shift: From Optimization to Satisfiability

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This is the world of **satisfiability**.

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Observation: in all of these, the answer is yes or no. If yes, we want a witness. If no, we want a proof. There is nothing to optimize.

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We need a systematic framework. That framework is **propositional logic** and **SAT**.

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Example: let a = "Alice goes," b = "Bob goes," c = "Carol goes."
"If Alice goes, then Bob does not go" becomes:

$$a \rightarrow \neg b$$

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Notice: every rule became a disjunction (OR of literals). This is not a coincidence.

CNF: Conjunctive Normal Form

Definitions

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A **clause** is a disjunction (OR) of literals: $(\neg a \vee \neg b)$.

A formula in **CNF** is a conjunction (AND) of clauses.

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Our party puzzle in CNF:

$$\underbrace{(\neg a \vee \neg b)}_{\text{rule 1}} \wedge \underbrace{(b \vee c)}_{\text{rule 2}} \wedge \underbrace{(a \vee b)}_{\text{rule 3}} \wedge \underbrace{(\neg a \vee \neg c)}_{\text{rule 4}}$$

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Key fact: every propositional formula can be converted to an equivalent CNF formula. So CNF is the “universal input format” for SAT solvers.

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Our CNF formula: $(\neg a \vee \neg b) \wedge (b \vee c) \wedge (a \vee b) \wedge (\neg a \vee \neg c)$.

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F	T	T	T	T	T	T	**Yes!**

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Two satisfying assignments. Both require: Alice stays home, Bob goes. Carol can go or not.

The SAT Problem

Boolean Satisfiability (SAT)

Input: a CNF formula with n variables and m clauses.
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Output: SAT + a satisfying assignment, or UNSAT.

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The question is: can we do better than brute force?

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The gap between worst-case theory and practical performance is *enormous*. Understanding why is one of the great mysteries of computer science. We will peek under the hood in Lecture 18.

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Variables	$x_i \in \{0, 1\}$ or $\mathbb{Z}$	$x_i \in \{\mathrm{T}, \mathrm{F}\}$
Constraints	Linear inequalities	Clauses (OR of literals)
Objective	Minimize $c^\top x$	None (feasibility only)
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Takeaway: SAT is not “better” or “worse” than IP. It is a different language. The right tool depends on what your constraints look like.

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You declare variables and constraints. Z3 searches for a satisfying assignment. You never write a search algorithm.

Z3 Basics: The Party Puzzle

Let's solve our party puzzle with Z3.

```
from z3 import *                                  #New!!

a = Bool('a')  # Alice goes?
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s = Solver()
s.add(Implies(a, Not(b)))     # rule 1: if Alice then not Bob
s.add(Implies(Not(b), c))     # rule 2: if not Bob then Carol
s.add(Or(a, b))               # rule 3: Alice or Bob
s.add(Or(Not(a), Not(c)))     # rule 4: not both Alice and Carol
```

Z3 Basics: The Party Puzzle

Let's solve our party puzzle with Z3.

```
from z3 import *                            #New!!

a = Bool('a')  # Alice goes?
b = Bool('b')  # Bob goes?
c = Bool('c')  # Carol goes?

s = Solver()
s.add(Implies(a, Not(b)))     # rule 1: if Alice then not Bob
s.add(Implies(Not(b), c))     # rule 2: if not Bob then Carol
s.add(Or(a, b))               # rule 3: Alice or Bob
s.add(Or(Not(a), Not(c)))     # rule 4: not both Alice and Carol

print(s.check())              # sat
print(s.model())              # [a = False, b = True, c = False]
```

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- `s.check()` returns one of three values:
  - `sat`: a satisfying assignment exists.
  - `unsat`: no assignment can satisfy all constraints.
  - `unknown`: Z3 could not determine (**very** rare for pure SAT).

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- `s.model()` returns the actual assignment (the "witness").
- We can query individual values: `s.model()[a]` returns `False`.

Key insight: compare this to Gurobi's workflow. In Gurobi: `m.optimize()`, then `x.X` to read values. In Z3: `s.check()`, then `s.model()` to read values.

What If It’s UNSAT?

Add a fifth rule: “Bob must not go.”

```
from z3 import *
set_param(proof=True) #Generate proof for unsatisfiability!

s = Solver()
s.add(Implies(a, Not(b)))      # rule 1
s.add(Implies(Not(b), c))     # rule 2
s.add(Or(a, b))               # rule 3
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s.add(Not(b))                 # rule 5: Bob does NOT go     #New
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if s.check() == unsat:
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print(s.check())   # unsat
if s.check() == unsat:
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```

Why? Rule 3 says $a \lor b$. Rule 5 says $\neg b$. So a must be true. But then rule 1 forces $\neg b$ (already satisfied) and rule 2 forces c. Now rule 4 says $\neg a \lor \neg c$, but a and c are both true. Contradiction.

Sudoku: The Poster Child for SAT

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5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
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No objective function. Pure feasibility. This is a SAT problem in disguise.

Encoding Sudoku in CNF

Variables: for each cell (i, j) and value $v \in \{1, \ldots, 9\}$, a **Boolean** variable:

$$x_{i,j,v} = \begin{cases} \text{True} & \text{if cell } (i,j) \text{ holds value } v \\ \text{False} & \text{otherwise} \end{cases}$$

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- **Exactly one value per cell:**
  - At least one: $(x_{i,j,1} \vee x_{i,j,2} \vee \cdots \vee x_{i,j,9})$ for all i, j.
  - At most one: $(\neg x_{i,j,v_1} \vee \neg x_{i,j,v_2})$ for every pair $v_1 \neq v_2$

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- **Clues:** if cell (i, j) is pre-filled with v, add the unit clause $(x_{i,j,v})$.

Every constraint is an OR of literals. This is **pure CNF**: ∼12,000 clauses, each with at most 9 literals.

Sudoku in Z3: Variables and the “Exactly One” Pattern

```
from z3 import *

# x[i][j][v] = True iff cell (i,j) holds value v+1
X = [[[Bool(f'x_{i}_{j}_{v}') for v in range(9)]
      for j in range(9)] for i in range(9)]
s = Solver()
```

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s = Solver()
def exactly_one(bools):
    """Encode 'exactly one is True' as pure CNF clauses."""
    s.add(Or(bools))                          # at-least-one clause
    for a in range(len(bools)):               # pairwise at-most-one
        for b in range(a+1, len(bools)):
            s.add(Or(Not(bools[a]), Not(bools[b])))
```

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    for a in range(len(bools)):                 # pairwise at-most-one
        for b in range(a+1, len(bools)):
            s.add(Or(Not(bools[a]), Not(bools[b])))
# Each cell holds exactly one value
for i in range(9):
    for j in range(9):
        exactly_one([X[i][j][v] for v in range(9)])
```

Sudoku in Z3: Rows, Columns, Boxes, and Clues

```python
# Each value appears exactly once per row
for i in range(9):
    for v in range(9):
        exactly_one([X[i][j][v] for j in range(9)])

# Each value appears exactly once per column
for j in range(9):
    for v in range(9):
        exactly_one([X[i][j][v] for i in range(9)])
```

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# Each value appears exactly once per row
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# Each value appears exactly once per column
for j in range(9):
    for v in range(9):
        exactly_one([X[i][j][v] for i in range(9)])

# Each value appears exactly once per 3x3 box
for bi in range(3):
    for bj in range(3):
        for v in range(9):
            box = [X[3*bi+di][3*bj+dj][v]
                   for di in range(3) for dj in range(3)]
            exactly_one(box)
```

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        for v in range(9):
            box = [X[3*bi+di][3*bj+dj][v]
                   for di in range(3) for dj in range(3)]
            exactly_one(box)

# Clues: assert the known value directly (unit clause)
clues = [(0,0,5),(0,1,3),(0,4,7),(1,0,6),(1,3,1),(1,4,9),(1,5,5), ...]
for (i, j, v) in clues:
    s.add(X[i][j][v-1])          # just one literal
```

Sudoku in Z3: Solving and Reading the Solution

```
if s.check() == sat:
    m = s.model()
    for i in range(9):
        row = [v+1 for j in range(9) for v in range(9)
               if is_true(m[X[i][j][v]])]
        print(' '.join(str(d) for d in row))
```

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if s.check() == sat:
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Output:

```
5 3 4 6 7 8 9 1 2
6 7 2 1 9 5 3 4 8
1 9 8 3 4 2 5 6 7
8 5 9 7 6 1 4 2 3
4 2 6 8 5 3 7 9 1
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```

Z3 solves this in around 46 milliseconds. We fed it ∼12,000 clauses over 729 Boolean variables. Trivial for a modern SAT solver.

The N-Queens Problem

Place n queens on an $n \times n$ chessboard so that no two queens attack each other.

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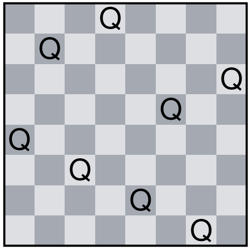

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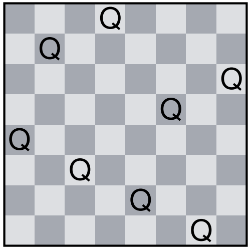

For $n = 8$, there are $\binom{64}{8} \approx 4.4$ billion ways to place 8 queens. Only 92 are valid.

N-Queens in Z3: Pure Boolean Encoding

$q_{i,j}$ = True iff there is a queen at row i, column j. That is $n^2 = 64$ Boolean variables.

```
from z3 import *
n = 8
Q = [[Bool(f'q_{i}_{j}') for j in range(n)] for i in range(n)]
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from z3 import *
n = 8
Q = [[Bool(f'q_{i}_{j}') for j in range(n)] for i in range(n)]
s = Solver()
for i in range(n):                                      # exactly one queen per row
    s.add(Or([Q[i][j] for j in range(n)]))              # at-least-one clause
    for j1 in range(n):                                 # pairwise at-most-one
        for j2 in range(j1+1, n):
            s.add(Or(Not(Q[i][j1]), Not(Q[i][j2])))
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        for j2 in range(j1+1, n):
            s.add(Or(Not(Q[i][j1]), Not(Q[i][j2])))
for j in range(n):                                  # at most one per column
    for i1 in range(n):
        for i2 in range(i1+1, n):
            s.add(Or(Not(Q[i1][j]), Not(Q[i2][j])))

for i1 in range(n):                                 #at most one per diagonal
    for j1 in range(n):
        for i2 in range(i1+1, n):
            for d in [j1+(i2-i1), j1-(i2-i1)]:     # both diags
                if 0 <= d < n:
                    s.add(Or(Not(Q[i1][j1]), Not(Q[i2][d])))
```

N-Queens: Solving

```python
print(s.check())    # sat
m = s.model()
for i in range(n):
    col = [j for j in range(n) if is_true(m[Q[i][j]])][0]
    print(f"Row {i}: column {col}")
# Row 0: column 3, Row 1: column 6, ...
```

N-Queens: Scaling Up

n	**Z3 solving time**
8	$<$ 0.01 seconds
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50	$\sim$ 1 seconds
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But: what if you wanted extra constraints? "Queens in rows 0 and 7 must be in adjacent columns." With Z3, you add two more lines. With a hand-written solver, you rewrite your solver.

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Takeaway: Z3 trades raw speed for modeling flexibility. When constraints are complex and evolving, that tradeoff is worth it.

Finding ALL Solutions

Sometimes we don't just want *a* solution, we want *all* of them.

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Z3 returns one model at a time. To enumerate, we add a **blocking clause** after each solution:

```
n = 5   # 5-queens (10 solutions exist)
Q = [[Bool(f'q_{i}_{j}') for j in range(n)] for i in range(n)]
s = Solver()
# ... (same row/column/diagonal constraints as before) ...
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s = Solver()
# ... (same row/column/diagonal constraints as before) ...
count = 0
while s.check() == sat:
    m = s.model()
    count += 1
    # Block this placement: negate at least one placed queen
    block = [Not(Q[i][j]) for i in range(n) for j in range(n)
             if is_true(m[Q[i][j]])]
    s.add(Or(block))                      # itself a clause!

print(f"Total solutions: {count}")
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    s.add(Or(block))                      # itself a clause!

print(f"Total solutions: {count}")
```

Key idea: the blocking clause is itself pure CNF: an OR of negated literals. We tell Z3 "not that exact placement" and ask again.