CS498: Algorithmic Engineering

Lecture 18: How SAT Solvers Actually Work

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Outline

Where We Left Off

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Today: we open the black box. How does a SAT solver actually work?

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The question is: can we do better?

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"Simplify": remove satisfied clauses, drop false literals from remaining clauses.
Still 2^n worst-case, but we prune early $\Rightarrow$ skip entire subtrees.

Unit Propagation: The Key Idea

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Key insight: UP *cascades*. Forcing x_3 might make another clause unit, which forces another variable, and so on.

Unit Propagation: Cascade Example

$$\underbrace{(x_1 \vee x_2)}_{C_1} \wedge \underbrace{(\neg x_1 \vee x_3)}_{C_2} \wedge \underbrace{(\neg x_2 \vee x_4)}_{C_3} \wedge \underbrace{(\neg x_3 \vee \neg x_4)}_{C_4} \wedge \underbrace{(x_2 \vee \neg x_3)}_{C_5}$$

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- C_2: $\neg x_1 = \text{True} \Rightarrow$ satisfied.
- $C_4 = (\neg x_3 \vee \cancel{\neg x_4})$ is unit. **Force** $x_3 = \texttt{False}$.

One decision triggered a cascade assigning *all* variables. SAT.

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Less impactful than UP in practice, but cheap and sometimes helpful.

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3. If no clauses left: return SAT.
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5. Pick an unassigned variable x.
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In words: propagate for free, then guess, then recurse. Dead end $\Rightarrow$ backtrack.

DPLL Walkthrough: SAT Instance

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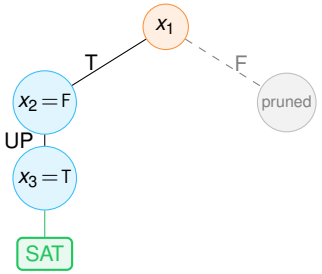

DPLL Walkthrough: UNSAT Instance

An unsatisfiable formula (4 variables, 8 clauses):

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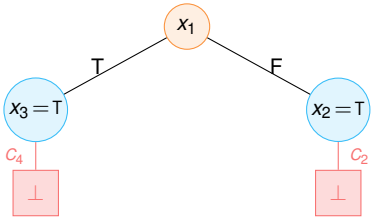

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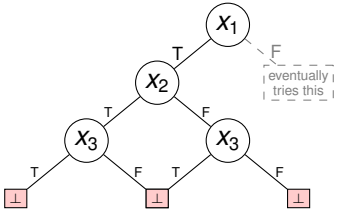
- Conflict at level 10: caused by x_1 and x_{10}, nothing else.
- DPLL backtracks to level 9. But x_9 was irrelevant. Same conflict. Again at 8, 7, 6, ...
- Result: 2^8 useless combinations of $x_2, \ldots, x_9$ before reaching the real culprit.

DPLL Makes the Same Mistake Over and Over



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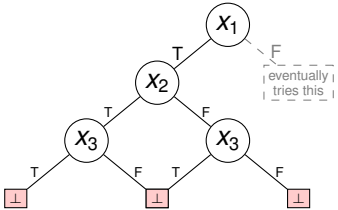

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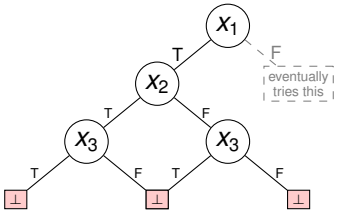

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The gap

DPLL does not learn *why* conflicts happen. It makes the same mistake over and over.

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This is exactly what **CDCL** (Conflict-Driven Clause Learning) does.

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1. **Implication Graph.** Track *why* each variable was set.
2. **Clause Learning.** When a conflict occurs, analyze the graph, derive a new clause that prevents this mistake forever.
3. **Non-chronological Backjumping.** Jump back to the *root cause*, not the most recent decision.

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L0	No decisions yet. Propagate original unit clauses (and learned unit clauses).
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- **Level 0** exists before any decision. It is the "ground truth."
- **Backjumping to level k:** undo all assignments from levels $> k$. Levels $0, \ldots, k$ stay intact.

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- **Decision nodes** (orange): variables set by choice. Labeled with decision level.
- **Propagation edges** (blue): clause C_j forced variable y, so draw edges from the other literals in C_j to y.
- **Conflict node** ($\perp$, red): when propagation forces contradictory values.

CDCL Example: The Formula

$$C_1 = (x_1 \vee x_4)$$
$$C_2 = (x_1 \vee \neg x_3 \vee \neg x_5)$$
$$C_3 = (\neg x_1 \vee x_2)$$
$$C_4 = (\neg x_1 \vee x_3 \vee x_5)$$
$$C_5 = (\neg x_2 \vee \neg x_4)$$
$$C_6 = (\neg x_2 \vee x_3)$$
$$C_7 = (\neg x_3 \vee x_4)$$
$$C_8 = (x_2 \vee \neg x_4 \vee x_5)$$

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$$C_5 = (\neg x_2 \vee \neg x_4) \qquad C_6 = (\neg x_2 \vee x_3)$$
$$C_7 = (\neg x_3 \vee x_4) \qquad C_8 = (x_2 \vee \neg x_4 \vee x_5)$$

5 variables, 8 clauses. Let's solve it with CDCL.

Step 1: Decide and Propagate

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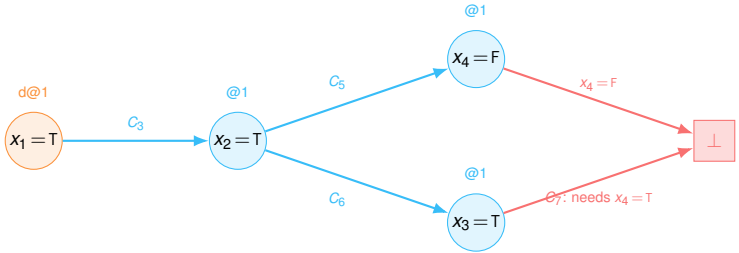
Conflict: step 2 forced $x_4 = \mathtt{F}$, but step 4 needs $x_4 = \mathtt{T}$.

Step 2: Draw the Implication Graph



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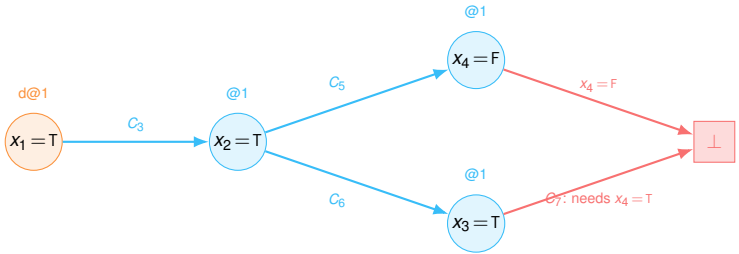

The graph shows: x_2 is the **fork point**. It forced $x_4 = \mathsf{F}$ (via C_5) and $x_3 = \mathsf{T}$ (via C_6), and x_3 then demanded $x_4 = \mathsf{T}$ (via C_7).

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2. x_3 was forced by $C_6 = (\neg x_2 \vee x_3)$. **Resolve** on x_3:

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Learned clause: $(\neg x_2)$. This clause is *permanently* added to the formula.

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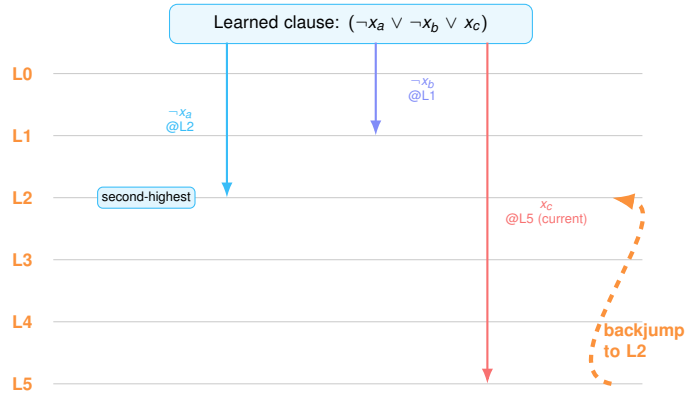
Takeaway: resolve backwards until one literal remains at the current level. That gives you the learned clause.

Where to Backjump: The General Rule



Backjump Rule

Find the **second-highest decision level** among all literals in learned clause. Backjump there. The clause is **unit** at that level, forcing the remaining literal.

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CDCL does:

- **Backjump to level 0.**
- $(\neg x_2)$ becomes unit.
- Force $x_2 = \texttt{F}$.

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All 8 clauses satisfied. **SAT:** $x_1 = \mathtt{F},\ x_2 = \mathtt{F},\ x_3 = \mathtt{F},\ x_4 = \mathtt{T},\ x_5 = \mathtt{T}$.

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Observation: one conflict, one learned clause, *zero* further decisions. UP did all the work.

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Key insight: CDCL might undo 9 decisions at once, because it *knows* the last 8 were irrelevant.

CDCL vs. DPLL: Summary

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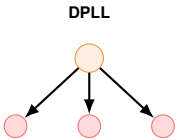

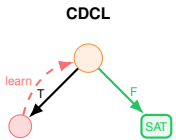

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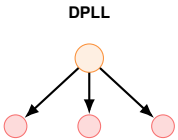

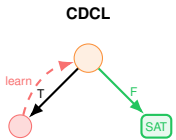

Takeaway: CDCL = DPLL + memory. It remembers its mistakes.

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2. **Restarts:** when to abandon and start over.

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Key insight: variables in recent conflicts are "hot." VSIDS focuses the solver on the active part of the problem. Decay ensures old conflicts fade.

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Takeaway: restarts + clause learning = explore broadly without losing knowledge.

DPLL vs. CDCL: Complete Comparison

Feature	DPLL (1962)	CDCL (1996+)
Core idea	Backtracking + UP	Backtracking + UP
Conflict response	Backtrack one level	Analyze, learn, backjump
Clause database	Static	Grows
Decision heuristic	Any	VSIDS
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The deal: CDCL adds complexity, but the payoff is **four orders of magnitude** in scalability.