CS498: Algorithmic Engineering

Lecture 19: SMT = SAT + Theories

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Outline

Where We Left Off

Lecture 17: we used Z3 to solve puzzles with **Bool** variables.

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So we understand SAT. The question is: what happens when the problem is not purely boolean?

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The gap

CDCL speaks boolean. Our problems speak integers, arrays, functions. How to bridge?

A Motivating Example

Suppose we want to solve:

$$x + 2y = 9, \quad x > 0, \quad y > 0, \quad x \leq y, \quad f(x) = f(y) \quad \forall f, \quad x \in \mathbb{Z} \quad y \in \mathbb{Z}.$$

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Keep the SAT engine for boolean reasoning, plug in **specialized solvers** for arithmetic, arrays, functions, etc. Let them collaborate.

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A theory solver = specialized engine that answers “is this conjunction of constraints feasible?” for a specific kind of constraint.

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A specialized feasibility checker for one kind of constraint.

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Also: **LRA** (Linear Real Arithmetic) — same symbols, variables over $\mathbb{R}$. Easier (LP instead of ILP).

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Implementation: union-find (disjoint set) data structure. Learn $a = b$? Union their sets. Check equality? Compare representatives.

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Key: reason about *relationships between calls* without knowing what functions do.

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3. **DPLL(T)**: tight collaboration. What Z3 does*.

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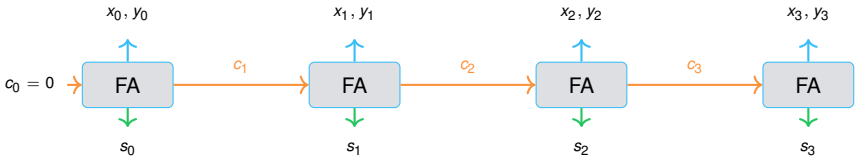
Result: a standard SAT problem. CDCL can solve it.

Bit-Blasting: What the Adder Looks Like

A **full adder** for one bit position:



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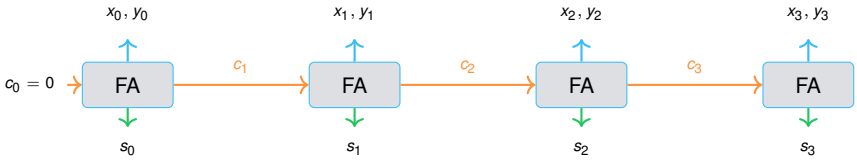

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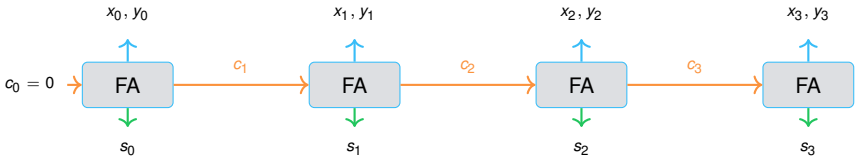

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4 bits: $\sim$20+ boolean variables just for addition. Now imagine 32- or 64-bit integers.

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But:

Bit-blasting *is* the method of choice for **bitvector** theories (fixed-width machine integers). For general integer arithmetic, we need something smarter.

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1. Boolean abstraction.

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Pure SAT. The solver has no idea what b_1 "means": just booleans and clauses.

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Does this conjunction have an integer solution? Hand it to the **LIA theory solver**.

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Key

SAT solver handled booleans (OR, AND). Theory solver handled arithmetic. Teamwork.

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Fix: generate a **blocking clause** ("theory lemma") to prevent this mistake.

Lazy Step 4: Blocking Clauses

$\{\neg b_1, b_2, b_3, b_4\}$ is infeasible. Block it:

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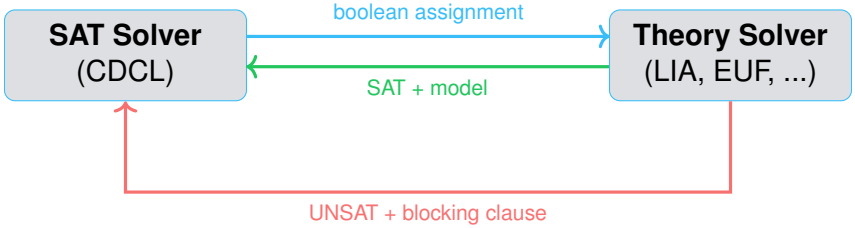
$b_3 = \mathsf{T}$, $b_4 = \mathsf{T} \Rightarrow$ lemma simplifies to b_1. Now $b_1 = \mathsf{T}$.
SAT solver finds $b_1 = b_2 = b_3 = b_4 = \mathsf{T}$, theory confirms. Done.

The Lazy Loop



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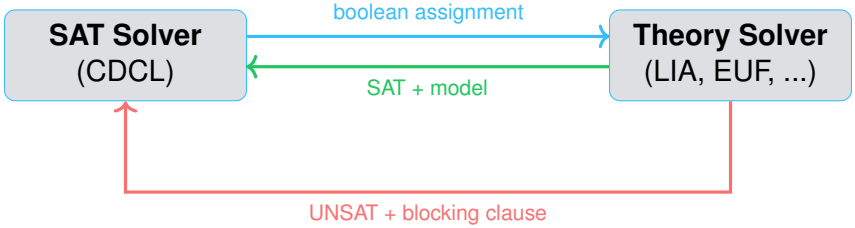

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Terminates: finitely many boolean assignments; each clause eliminates ≥ 1.

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Yes. $\Rightarrow$ DPLL(T).

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Done in one pass. No round trips.

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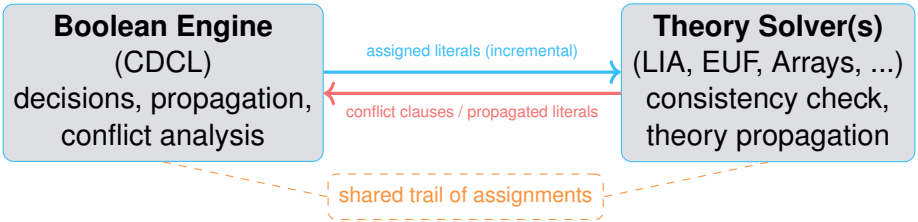

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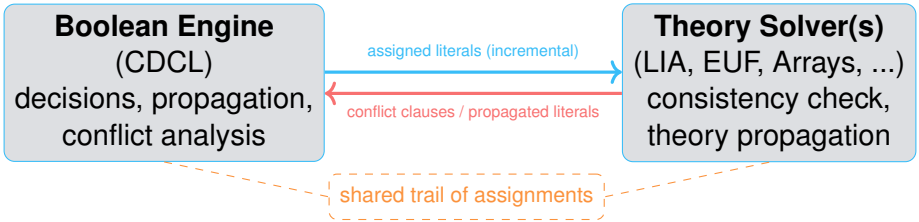

Tightly integrated. Information shared *during* search, not just at the end.

Three Approaches Compared

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DPLL(T) dominates. Early conflicts + theory propagation $\Rightarrow$ much less search. Architecture behind Z3, CVC5, Yices.

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$$1000S+100E+10N+D+1000M+100O+10R+E = 10000M+1000O+100N+10E+$$

Plus: $S \neq 0$, $M \neq 0$ (no leading zeros), all digits distinct. Pure LIA.

SEND + MORE = MONEY: Z3 Code

```
from z3 import *

S, E, N, D, M, O, R, Y = Ints('S E N D M O R Y')
s = Solver()

digits = [S, E, N, D, M, O, R, Y]
s.add([And(d >= 0, d <= 9) for d in digits])   # each digit 0-9
s.add(Distinct(digits))                         # all different
s.add(S != 0, M != 0)                          # no leading zeros
```

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# The arithmetic constraint                    #New!!
send  = 1000*S + 100*E + 10*N + D
more  = 1000*M + 100*O + 10*R + E
money = 10000*M + 1000*O + 100*N + 10*E + Y

s.add(send + more == money)
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s.add(send + more == money)

print(s.check())    # sat
m = s.model()
for v in digits:
    print(f"{v} = {m[v]}", end="  ")
# S = 9  E = 5  N = 6  D = 7  M = 1  O = 0  R = 8  Y = 2
# 9567 + 1085 = 10652
```

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from z3 import *

x, y, z = Ints('x y z')
f = Function('f', IntSort(), IntSort())   #New!! uninterpreted function

s = Solver()
s.add(f(x) == y)
s.add(x == z)
s.add(f(z) != y)   # assert the negation: can f(z) differ from y?
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print(s.check())    # unsat
# No! By congruence: x = z implies f(x) = f(z).
# Since f(x) = y, we must have f(z) = y.
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We never defined f. Z3 used only the congruence axiom: EUF solver at work.

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Key axiom: $\text{Select}(\text{Store}(a, i, v), i) = v$.

```
from z3 import *

a = Array('a', IntSort(), IntSort())   #New!! array: int -> int

# Write 42 at index 3, then read index 3
b = Store(a, 3, 42)
print(simplify(Select(b, 3)))   # 42
```

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Two operations:

- `Store(a, i, v)`: new array with $a[i] = v$.
- `Select(a, i)`: read $a[i]$.

Key axiom: $\text{Select}(\text{Store}(a, i, v), i) = v$.

```
from z3 import *

a = Array('a', IntSort(), IntSort())   #New!! array: int -> int

# Write 42 at index 3, then read index 3
b = Store(a, 3, 42)
print(simplify(Select(b, 3)))   # 42
# Harder: find an array where a[0] + a[1] + a[2] = 10
# and all entries are between 1 and 5
s = Solver()
for i in range(3):
    s.add(Select(a, i) >= 1, Select(a, i) <= 5)
s.add(Select(a, 0) + Select(a, 1) + Select(a, 2) == 10)

print(s.check())    # sat
print(s.model())    # e.g., a[0]=5, a[1]=4, a[2]=1
```

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Specialization wins.

What If Your Formula Mixes Theories?

$$f(x) = f(y) \quad \wedge \quad x + 1 = y \quad \wedge \quad x > 0 \quad \wedge \quad \text{Select}(a, x) \neq \text{Select}(a, y)$$

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No single solver handles all of this. How do we combine them?

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- LIA deduces $x \neq y$, shares with EUF.
- EUF: $f(x) = f(y)$ still consistent (f could map different inputs to same output). No contradiction from EUF alone.

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Takeaway

Z3 = team of specialists, coordinated by a SAT engine. You just state the problem.