CS498: Algorithmic Engineering

Lecture 20: SMT Applications: Puzzles, Verification, and Proof

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Week 11 – 04/02/2026

Outline

Today's Roadmap

1. **Encoding tricks:** cardinality, symmetry breaking, choosing the right theory.

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Theme: same negation trick, four very different applications.

The “At Most k” Problem

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- At-least-one: `Or`$(x_1, \ldots, x_n)$. One clause.
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$\Rightarrow$ Auxiliary variables trade space for an *exponential* reduction in clauses.

Z3's Built-in: PbLe / PbGe

Implementing sequential counters by hand is tedious. Z3 wraps this into one function:

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from z3 import *
xs = [Bool(f'x_{i}') for i in range(20)]
s = Solver()
s.add(PbLe([(x, 1) for x in xs], 5))    # at most 5 True      #New!!
s.add(PbGe([(x, 1) for x in xs], 3))    # at least 3 True     #New!!
# PbEq([(x, 1) for x in xs], 4)         # exactly 4 True
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Encoding lesson

When you need "at most k" for $k > 1$, Use `PbLe/PbGe/PbEq`.

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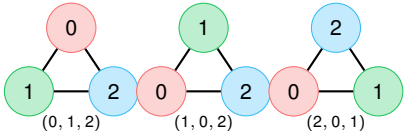

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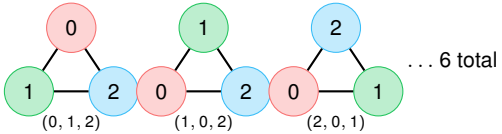

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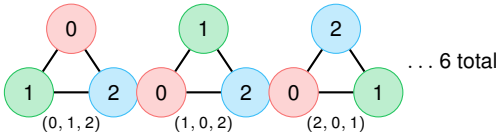

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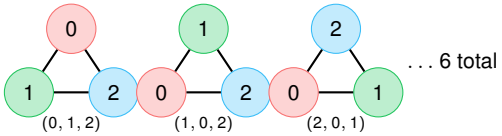

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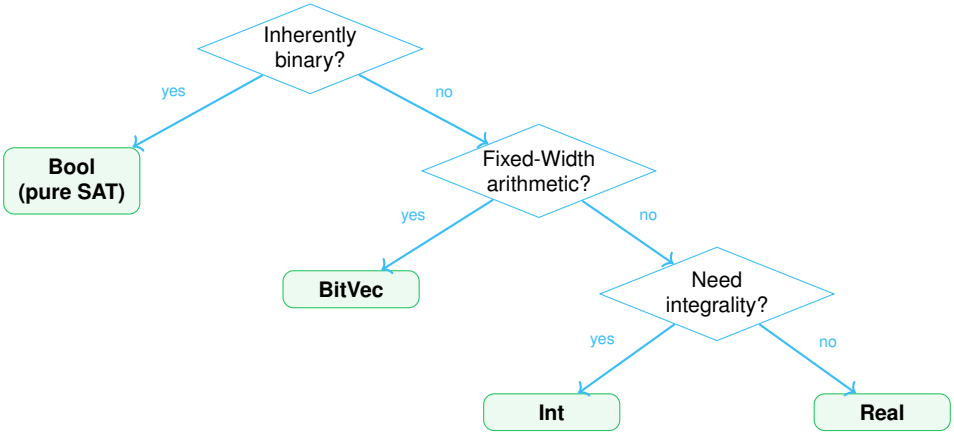
Now only $(0, 1, 2)$ survives. $\sim 9\times$ speedup for $k = 5$; over $100\times$ for $k = 10$.

Choosing the Right Encoding

Encoding	Best when	Watch out for
Bool (SAT)	Inherently binary choices.	Blows up for counting/arithmetic.
Int (LIA)	Counting, permutations,	Nonlinear $\Rightarrow$ undecidable (NIA).
BitVec	Hardware, software, crypto, fixed-width arithmetic.	Bit-blasting large widths is slow.
Real (LRA/NRA)	Continuous quantities. Nonlinear OK (decidable).	Cannot express integrality.

Decision Flowchart



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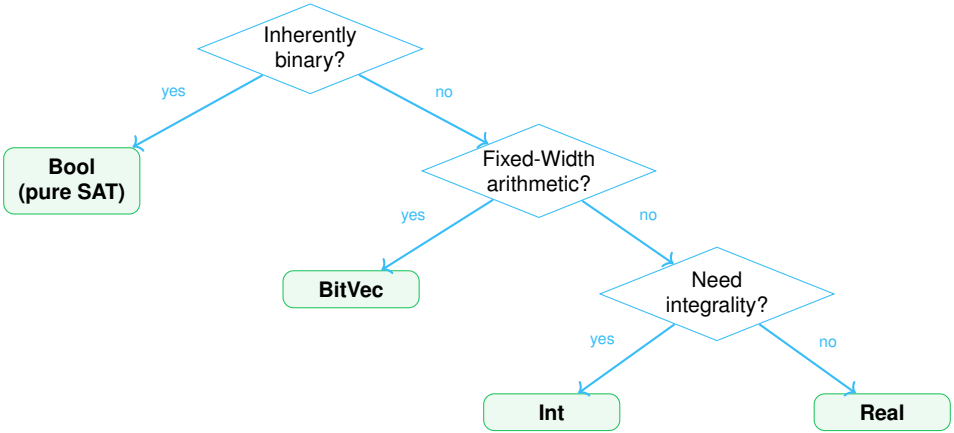

This is a starting point. In practice, the best encoding often **mixes** theories: Bool grid + Int positions (Nonograms), BitVec for machine arithmetic (verification).

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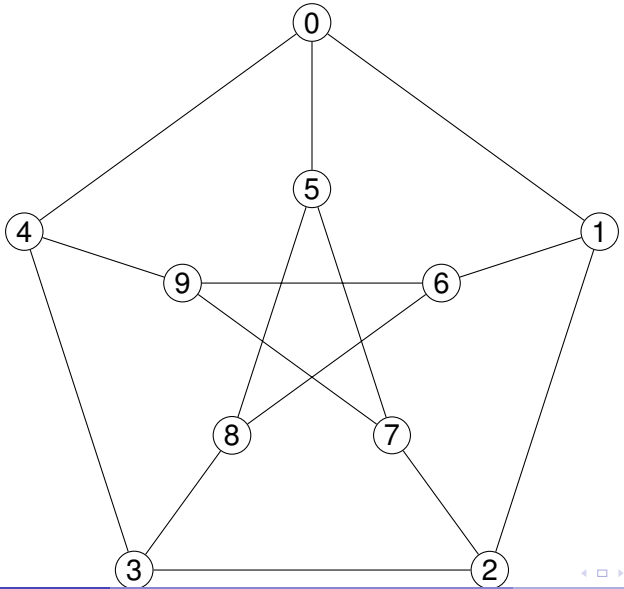
Encoding choices:

- **Int** variable per vertex: $c_i \in \{0, \ldots, k-1\}$.
- Edge constraint: $c_u \neq c_v$ for each $(u, v) \in E$.
- Symmetry breaking: fix $c_0 = 0$, and $c_i \leq i$ (from our encoding tricks).

Sample Graph



Graph Coloring: Z3 Code

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from z3 import *
# Petersen graph: 10 vertices, 15 edges, chromatic number = 3
edges = [(0,1),(0,4),(0,5),(1,2),(1,6),(2,3),(2,7),
         (3,4),(3,8),(4,9),(5,7),(5,8),(6,8),(6,9),(7,9)]
n, k = 10, 3
c = [Int(f'c_{i}') for i in range(n)]

s = Solver()
for i in range(n):
    s.add(c[i] >= 0, c[i] < k)
for u, v in edges:
    s.add(c[u] != c[v])                    # adjacency
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print(s.check())  # sat
m = s.model()
print([m[c[i]].as_long() for i in range(n)])
# e.g. [0, 1, 2, 1, 2, 2, 2, 1, 0, 0]
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The Petersen graph is a classic: 3-colorable but not 2-colorable. Z3 finds a valid 3-coloring instantly.

Graph Coloring: Proving a Lower Bound

Can we 2-color the Petersen graph?

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$\Rightarrow$ SMT can *prove* graph properties, not just find solutions.

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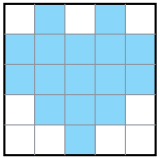

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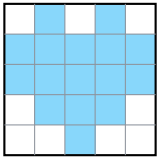

Observation: cells are binary, so Bool variables are natural. But the run-length constraints are the challenge.

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- Runs are ordered with gaps: $s_{j+1} \geq s_j + r_j + 1$ (at least one empty cell between runs).
- A cell (i, c) is filled iff it falls within some run: $\texttt{grid}[i][c] \Leftrightarrow \bigvee_j (s_j \leq c < s_j + r_j)$.

Nonogram: Z3 Code

```
from z3 import *

rows_clues, cols_clues = [[1,1],[5],[5],[3],[1]], [[2],[4],[4],[4],[2]]
R, C = len(rows_clues), len(cols_clues)
grid = [[Bool(f'g_{i}_{j}') for j in range(C)] for i in range(R)]
s = Solver()
def add_line_constraints(cells, clues, tag):
    n = len(cells)
    starts = [Int(f'{tag}_s{k}') for k in range(len(clues))]
    for k, r in enumerate(clues):
        s.add(starts[k] >= 0, starts[k] + r - 1 < n)         # fits in line
        if k > 0: s.add(starts[k] >= starts[k-1] + clues[k-1] + 1)
    for c_idx in range(n):
        in_some_run = Or([And(starts[k] <= c_idx, c_idx < starts[k] + clues[k])
                          for k in range(len(clues))])
        s.add(cells[c_idx] == in_some_run)

for i in range(R):
    add_line_constraints(grid[i], rows_clues[i], f'r{i}')
for j in range(C):
    add_line_constraints([grid[i][j] for i in range(R)], cols_clues[j], f'c{j}')
```

Nonogram: Solution

```
print(s.check())  # sat
m = s.model()
for i in range(R):
    row = ''.join('#' if is_true(m[grid[i][j]]) else '.'
                  for j in range(C))
    print(row)

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$\Rightarrow$ A mixed Bool + Int encoding. Each theory handles what it does best.

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Problem: you have two Boolean circuits (or two implementations of the same function). Are they equivalent on all inputs?

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Key insight: instead of proving equivalence (hard, universal quantifier), we search for a *difference* (easy, existential quantifier). UNSAT = proof of equivalence.

This is the same "proof by contradiction" pattern used in formal verification.

Circuit Equivalence: Z3 with BitVec

```
from z3 import *

x = BitVec('x', 32)  #New: 32-bit

# Circuit 1: clever bit-trick to check if x is a power of 2
f = If(x != 0, (x & (x - 1)) == 0, BoolVal(False))  # x & (x-1) == 0

# Circuit 2: naive approach
g = Or([x == (1 << i) for i in range(31)])

s = Solver()
s.add(f != g)                                        #the negation trick
result = s.check()
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print(result)  # sat -- they differ!

if result == sat:
    m = s.model()
    val = m[x].as_long()
    print(f"Counterexample: x = {val}")          # x = 2147483648 (2^31)
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Z3 found the bug: the naive circuit forgets 2^{31} (it only checks $i < 31$). BitVec makes this precise: fixed-width semantics catch overflow and boundary bugs.

Circuit Equivalence: Proving Correctness

```
from z3 import *

x = BitVec('x', 8)
#Compute popcount(x) mod 2 (parity of number of set bits in x)
# Method 1: fold parity down to the last bit
f = x
f = f ^ (f >> 1)
f = f ^ (f >> 2)
f = f ^ (f >> 4)
parity1 = f & 1

# Method 2: XOR the bits directly
parity2 = ((x >> 0) & 1) ^ ((x >> 1) & 1) ^ ((x >> 2) & 1) ^ ((x >> 3) & 1) \
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Two genuinely different implementations (XOR-fold vs. bit-by-bit). Z3 proves they agree on all $2^8 = 256$ inputs. Scale to 64 bits: still instant, because bit-blasting creates a compact SAT problem.

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- Sum $= 3{,}500{,}000{,}000 > 2^{31} - 1 = 2{,}147{,}483{,}647$.

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No overflow: `high - low` fits in a 32-bit signed int when $0 \leq$ `low` $\leq$ `high`.

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- The solver searches all 2^{64} possible (low, high) pairs instantly.

Key insight: `Int` would miss this: mathematical integers don't overflow. `BitVec(32)` models **exact machine arithmetic**.

Finding the Overflow: Z3 Code

```
from z3 import *

lo = BitVec('lo', 32)
hi = BitVec('hi', 32)

s = SolverFor("QF_BV")
s.add(lo >= 0)                    # signed comparison
s.add(hi >= 0)                    # signed comparison
s.add(lo <= hi)                   # signed comparison
s.add((lo + hi) < 0)              # wrapped sum looks negative (signed)
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print(s.check())   # sat -- overflow exists!
m = s.model()
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print(f"hi = {m[hi].as_signed_long()}") # 2147483647
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SAT. Z3 found concrete values where `lo + hi` wraps negative. This is a real bug that crashed Java programs on large arrays.

Verifying the Fix

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Two queries, two results:

- `(lo + hi) / 2`: SAT $\Rightarrow$ overflow bug exists.
- `lo + (hi - lo) / 2`: UNSAT $\Rightarrow$ fix verified for all inputs.

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The lesson

A 9-year-old bug in a JDK standard library, invisible to billions of test runs, is found in milliseconds by a 10-line Z3 script.

The Corner-Spheres Construction

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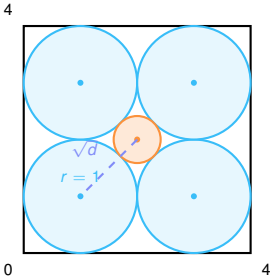
In 2D: 4 corner spheres, 1 small central sphere. Everything fits comfortably inside.



The Paradox: Central Sphere Radius

Distance from center $(2, \ldots, 2)$ to any corner center (e.g. $(1, 1, \ldots, 1)$):

$$\sqrt{(2-1)^2 + (2-1)^2 + \cdots} = \sqrt{\underbrace{1 + 1 + \cdots + 1}_{d}} = \sqrt{d}$$

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The cube face is at distance 2 from the center (e.g., coordinate 0 is at distance 2 from coordinate 2).

The Paradox: Central Sphere Radius

Distance from center $(2, \ldots, 2)$ to any corner center (e.g. $(1, 1, \ldots, 1)$):

$$\sqrt{(2-1)^2 + (2-1)^2 + \cdots} = \sqrt{\underbrace{1 + 1 + \cdots + 1}_{d}} = \sqrt{d}$$

Central sphere is tangent $\Rightarrow$ its radius is: $R = \sqrt{d} - 1$

The cube face is at distance 2 from the center (e.g., coordinate 0 is at distance 2 from coordinate 2).

Conjecture to verify: Is the sphere always inside the box for all dimensions? can Z3 *prove* this with exact arithmetic?

Z3 Code: Corner Spheres

```
from z3 import *

def corner_spheres(d):
    s = Solver()
    R = Real('R')
    p = [Real(f'p_{i}') for i in range(d)]           #point on sphere

    # Central sphere radius: (R+1)^2 = d  (avoids sqrt)
    s.add((R + 1) * (R + 1) == d)
    s.add(R >= 0)

    # p is on the central sphere: sum (p_i - 2)^2 = R^2
    s.add(Sum([(p[i] - 2) * (p[i] - 2) for i in range(d)]) == R * R)

    # p is outside the cube [0,4]^d
    s.add(Or([Or(p[i] < 0, p[i] > 4) for i in range(d)]))

    return s.check()
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- `Real` variables: exact algebraic numbers, no floating-point error.
- $(R+1)^2 = d$ encodes $R = \sqrt{d} - 1$ without needing a square root function.

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d =  4: unsat          d =  9: unsat    <-- R = 2 exactly (touches)
d =  5: unsat          d = 10: sat      <-- R = 2.16... (escapes!)
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Z3 uses **exact algebraic numbers** (NRA theory). No floating-point approximation. This is a rigorous mathematical proof, not a numerical simulation.

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The lesson

SMT is not just for puzzles and programs, it can verify mathematical claims with full rigor, answering questions where human geometric intuition fails.