CS498: Algorithmic Engineering

Lecture 24: Sequence Modeling, Tokenization & Attention

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Week 14

Outline

CNNs Are Great for Images. But...

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A CNN with kernel size 3 cannot connect word 1 to word 50 without many layers.

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Applications:

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Applications:

- Text generation: ChatGPT, Claude, Gemini
- Autocomplete: GitHub Copilot, Gmail Smart Compose
- Translation, summarization, code generation

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N-grams memorize co-occurrences but cannot generalize.

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PPL	Meaning
$\|V\| = 50{,}000$	Random guessing
≈ 20–30	Good language model
1	Perfect prediction

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Two questions remain:

1. How do we represent text as numbers? → Tokenization
2. What architecture processes the result? → Attention

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- This raw text needs to become a sequence of integers (we're doing sequence modeling!)

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Common words stay whole. Rare words split into known pieces. Tunable vocab size.

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All modern LLMs use subword tokenization.

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>>> list("hello".encode("utf-8"))
[104, 101, 108, 108, 111]          # 5 bytes for 5 chars

>>> list("こんにちは".encode("utf-8"))
[227, 129, 147, 227, 130, 147, ...] # 15 bytes for 5 chars
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Key insight: a byte vocabulary of just 256 tokens can represent any text in any language.

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This is exactly what Byte Pair Encoding (BPE) Tokenizer does.

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Greedy compression: each step maximally compresses the corpus by removing the most frequent pair.

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This table fits in memory. Now run BPE on this compact table.

Pre-tokenization in Code

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>>> import re
>>> PAT = r"'s|'t|'re|'ve|'m|'ll|'d| ?\w+| ?[^\s\w]+|\s+(?!\S)|\s+"
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Now we can run BPE on this compact table instead of the raw corpus.

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Count all adjacent pairs, weighted by frequency.

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(d,e): 3	(e,s): 9	(s,t): 9
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Vocabulary after 3 merges: {l, o, w, e, r, i, d, s, t, n} $\cup$ {st, est, lo}

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Decoding: IDs $\rightarrow$ byte sequences $\rightarrow$ concatenate $\rightarrow$ UTF-8 decode. No ambiguity.

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Takeaway: the byte-level starting point means BPE handles any language. Quality depends on how much of that language is in the training corpus.

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100,256 vocabulary tokens. <|endoftext|> gets its own fixed ID (100257). You can inspect any tokenizer this way: pip install tiktoken.

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Now we can convert any text to integer sequences. What architecture processes them?

From Token IDs to Vectors

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Now we have: text $\rightarrow$ token IDs $\rightarrow$ sequence of vectors in $\mathbb{R}^d$.

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This is attention.

The "it" Example

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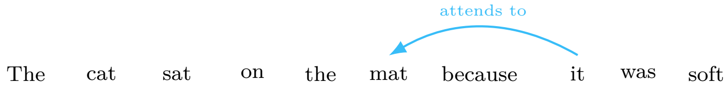

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The model needs "it" at position 8 to attend strongly to "mat" at position 6. The relevant word could be anywhere in the sentence. Attention lets each token decide, based on content, what to focus on.

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Core idea: dynamically weight inputs based on content.

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But not equally. A noun should attend more to other nouns than to verbs.

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"cat" has highest score with "cat" (1.0) and second highest with "mat" (0.5). Nouns match nouns.

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Weights α_{ij}	cat	sat	mat
cat →	0.51	0.19	0.31
sat →	0.19	0.51	0.31
mat →	0.51	0.19	0.31

Step 3: Softmax → Weights

Apply softmax to each row to get probabilities.

Row for "cat": scores $[1.0, 0.0, 0.5]$

$$\exp([1.0, 0.0, 0.5]) = [2.72, 1.00, 1.65], \quad \text{sum} = 5.37$$

$$\text{weights} = [2.72/5.37,\ 1.00/5.37,\ 1.65/5.37] \approx [\mathbf{0.51},\ 0.19,\ 0.31]$$

All three rows:

Weights α_{ij}	cat	sat	mat
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"cat" attends 51% to itself, 31% to "mat", 19% to "sat". Nouns attend to nouns.

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Each token's output is enriched with info from tokens it attends to.

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No magic. Matrix multiplication and a softmax is what gave us the AI boom :D.

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Comparing the approaches:

	MLP	CNN	Attention
Weights	Fixed	Fixed	Input-dependent
Range	None	Local	Global
Steps to connect	N/A	$O(T/k)$	$O(1)$
Cost per layer	$O(Td^2)$	$O(Tkd)$	$O(T^2 d)$

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Attention: same W_Q, W_K, W_V at every position, works for any T, parameters scale as $O(d^2)$ not $O(T^2d^2)$.

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Analogous to CNNs: early layers detect edges, later layers detect objects. Attention builds a hierarchy of relational patterns rather than spatial ones.

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Token 1 only sees itself. Token 2 sees tokens 1–2. Token 3 sees all.

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Next lecture: $\sqrt{\mathrm{d_k}}$ scaling, multi-head attention, positional encoding, full transformer.