CS498: Algorithmic Engineering

Lecture 3: Sensitivity Analysis & Network Models

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Outline

Theme of Today

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- Simplex walks along vertices (BFS).

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- Network flow models as a structured LP family where these ideas are very tangible.

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Goal: Use LP duals to answer: *"If I loosen this constraint a bit, how much better can I do?"*

Warm-up: Single-Constraint Example

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Dual:

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Dual optimal $y^* = 3$ is exactly **"$3 per extra unit of the constraint RHS."**

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Primal LP:

$$\begin{aligned} \max \quad & 5x \\ \text{s.t.} \quad & 2x \leq 40 \quad \text{(labor: 40 hours)} \\ & 1x \leq 30 \quad \text{(raw: 30 kg)} \\ & x \geq 0 \end{aligned}$$

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Constraints imply $x \leq 20$ and $x \leq 30 \Rightarrow x^* = 20$.

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$$z_P^* = 5 \cdot 20 = 100.$$

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Labor is binding, raw is slack.

Dual of the 2-Resource Example

Primal:

$$\max 5x \quad \text{s.t.} \begin{cases} 2x \leq 40 & \text{(labor)} \\ 1x \leq 30 & \text{(raw)} \\ x \geq 0 \end{cases}$$

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Introduce dual variables:

$$y_1 \geq 0 \text{ for labor}, \quad y_2 \geq 0 \text{ for raw}.$$

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Dual:

$$\begin{aligned} \min \quad & 40y_1 + 30y_2 \\ \text{s.t.} \quad & 2y_1 + 1y_2 \ \ge \ 5 \quad \text{(one constraint per primal variable)} \\ & y_1, y_2 \ge 0 \end{aligned}$$

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Optimum at $y_1^* = 2.5$, $y_2^* = 0$.

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Check against the primal: increase labor from 40 to 41.

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From the dual solution:

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Check against the primal: increase labor from 40 to 41.

- New labor constraint: $2x \leq 41 \Rightarrow x \leq 20.5$.

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From the dual solution:

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- New labor constraint: $2x \leq 41 \Rightarrow x \leq 20.5$.
- Raw constraint: $x \leq 30$ still non-binding.

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- New labor constraint: $2x \leq 41 \Rightarrow x \leq 20.5$.
- Raw constraint: $x \leq 30$ still non-binding.
- New optimum: $x' = 20.5$, $z' = 5 \cdot 20.5 = 102.5$.

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$$\Delta z_P^* = 102.5 - 100 = 2.5 \approx y_1^*.$$

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Raw is slack, so $y_2^* = 0$: extra raw does not improve the optimum.

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Only the binding labor constraint has nonzero shadow price.

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- y_i is the **marginal value** (shadow price) of resource i.
- It answers: **"How much does the optimal objective change if I get one more unit of this resource?"**

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Shadow prices light up exactly the bottleneck constraints.

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print("Primal solution:")
for p in products: print(p, x[p].X)

print("\nShadow prices:")
for r in resources: print(r, cons[r].Pi)
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- Primal: some optimal $(x^*_{\text{standard}}, x^*_{\text{deluxe}}) = (6, 8)$.
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- Raw has slack $\Rightarrow$ $y_{\text{raw}} = 0$ (extra raw is locally worthless).

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- CPU dual (y_{cpu}) = 0.05
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- **Buy GPUs**: each extra GPU-hour is worth roughly $50 of objective.

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Shadow prices = **shopping list** for infrastructure.

Shadow Prices for Variable Bounds in Gurobi

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Gurobi stores their shadow prices as **reduced costs**:

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- If x_j is at its **upper bound**, `x.RC` is the shadow price of $x_j \leq \mathrm{UB}_j$.

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- If x_j is at its **upper bound**, `x.RC` is the shadow price of $x_j \leq \mathrm{UB}_j$.
- If x_j is at its **lower bound**, `x.RC` is the shadow price of $x_j \geq \mathrm{LB}_j$.

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- How much water can we send from a reservoir to a city?
- What is the cheapest way to route water, if different pipes have different energy costs?

Flows: Think Water in Pipes

To introduce network flows, imagine:

- Nodes = junctions in a water network.
- Pipes = directed connections from one junction to another.
- At each pipe we can send some amount of water per minute.

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This intuition transfers directly to **cars on roads**, **packets on links**, **electricity on lines**, . . .

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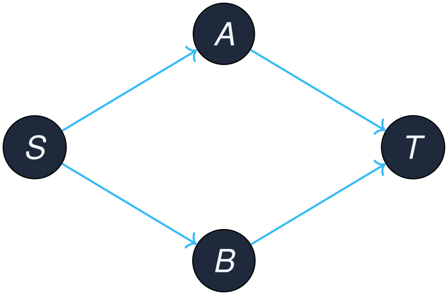

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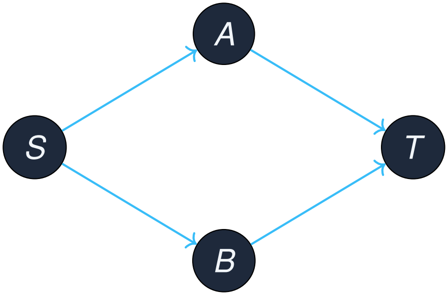

Goal (informally): pick flows on arcs to move "stuff" from S to T respecting the network.

Flows and Capacities

For each arc $(u, v) \in E$ we define:

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"The pipe can carry at most u_{uv} units per unit time."

Flow Conservation at Intermediate Nodes

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- Sum of flows **leaving** *n* minus sum of flows **entering** *n* is zero.
- What comes in must go out.

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Positive b_n: net outflow (supply). Negative b_n: net inflow (demand).

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We want to send the required flow **with minimum total cost.**

Minimum-Cost Flow as an LP

Put it all together:

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This is a standard LP with a lot of structure.

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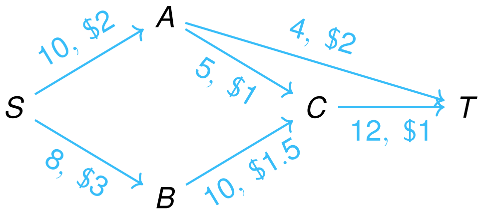
Gridlock LP Formulation

Decision variables: $f_{uv} \geq 0$ for each arc (u, v).



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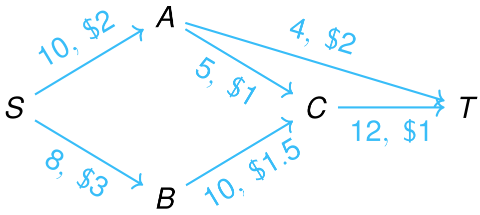

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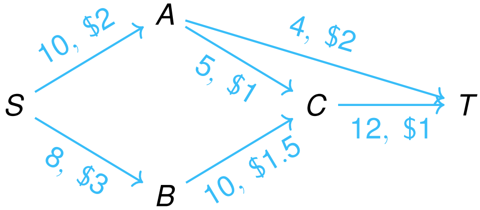

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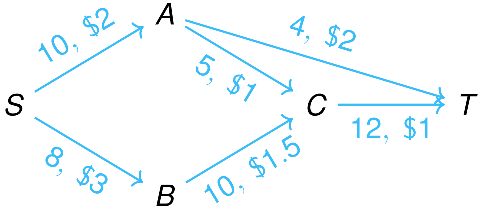

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Flow conservation:

$$\text{At } S: \quad f_{SA} + f_{SB} - 0 = 10,$$
$$\text{At } A: \quad f_{AC} + f_{AT} - f_{SA} = 0,$$
$$\text{At } B: \quad f_{BC} - f_{SB} = 0, \ldots$$

Solving Gridlock in Gurobi

```
import gurobipy as gp
import numpy as np

arcs = [('S','A',10,2.0), ('S','B',8,3.0), ('A','C',5,1.0),
    ('B','C',10,1.5), ('A','T',4,2.0), ('C','T',12,1.0)]

nodes = ['S', 'A', 'B', 'C', 'T']
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m.optimize()
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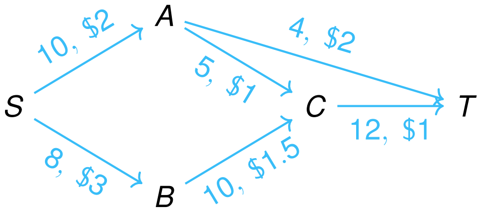
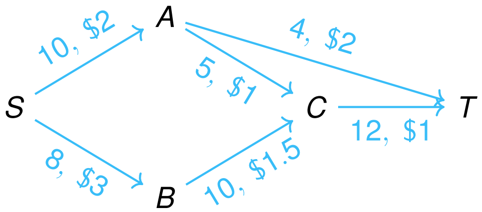
print(f"Optimal Cost: {m.ObjVal:.2f}")
for (u, v), var in flow.items():
    if not np.isclose(var.X, 0):
        print(f"  {u} -> {v}: {var.X:.1f}")
```



Baseline Gridlock Results

Optimal Flows:

$$f_{S\to A} = 9, \quad f_{S\to B} = 1, \quad f_{A\to C} = 5$$
$$f_{A\to T} = 4, \quad f_{B\to C} = 1, \quad f_{C\to T} = 6.$$



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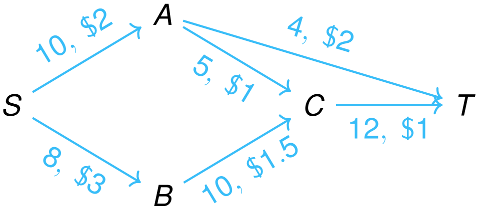
Total Cost: \$41.50



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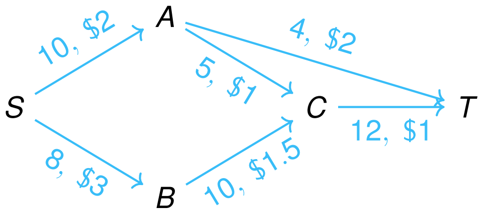
Cost Breakdown:

- $S \to A$: $9 \times 2.0 = 18.0$
- $S \to B$: $1 \times 3.0 = 3.0$
- $A \to C$: $5 \times 1.0 = 5.0$ (at capacity!)
- $A \to T$: $4 \times 2.0 = 8.0$ (at capacity!)
- $B \to C$: $1 \times 1.5 = 1.5$
- $C \to T$: $6 \times 1.0 = 6.0$



Shadow Prices on Capacities from Gurobi (Complementary Slackness)

Access capacity-related dual info in Gurobi:

```
print("\nShadow prices on capacities (from reduced costs):")
for u, v, cap, cost in arcs:
    var = flow[(u, v)]
    # For a binding upper bound, var.RC encodes a shadow price
    if np.isclose(var.X, cap):  # at (or very near) capacity
        print(f"  {u}->{v}: pi  {var.RC:.2f}")
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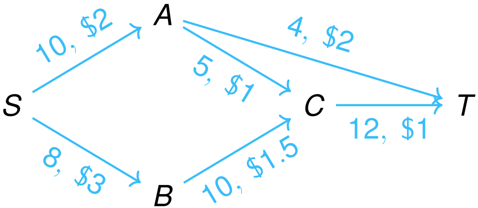
Interpretation: Adding 1 unit of capacity on $A \to C$ or $A \to T$ would *reduce* total cost by \$1.50.

Why Shadow Price ≈ -1.50?

Intuition: Look at competing routes.



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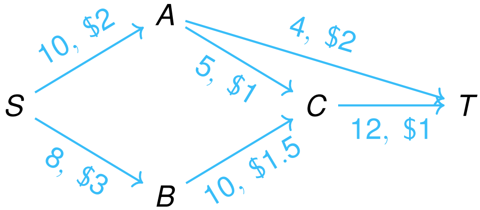
Cheap routes via A:

- $S \to A \to C \to T$:
  $2.0 + 1.0 + 1.0 = 4.0$/unit.
- $S \to A \to T$: $2.0 + 2.0 = 4.0$/unit.
- Both saturated.



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Next-best detour:

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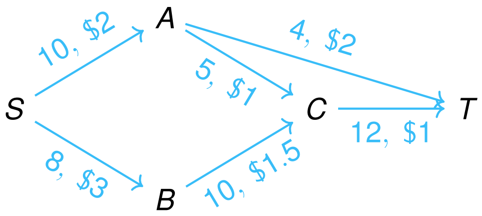
- Penalty: $5.5 - 4.0 = 1.5$ per unit.



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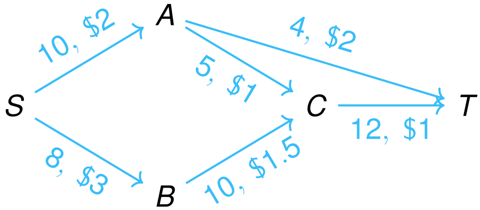
If we add 1 unit of capacity on $A \to C$, we can reroute 1 unit from the expensive detour to the cheap path, saving \$1.5.

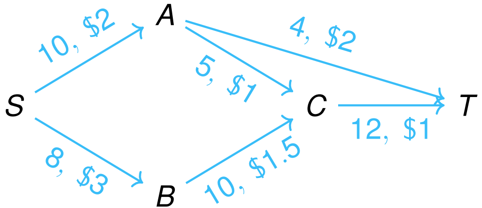


Scenario Analysis: Closing $A \to C$

What if arc $A \to C$ is blocked (capacity $= 0$)?

```
# Close A->C by setting capacity to 0 and re-optimizing
flow['A', 'C'].UB = 0
m.optimize()

print(f"New Cost after closing A->C: {m.ObjVal:.2f}")
```



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What if arc $A \to C$ is blocked (capacity $= 0$)?

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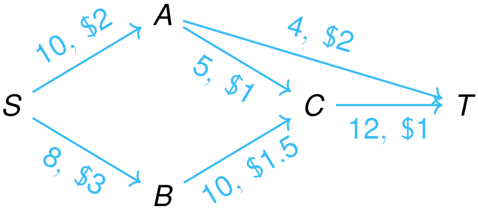

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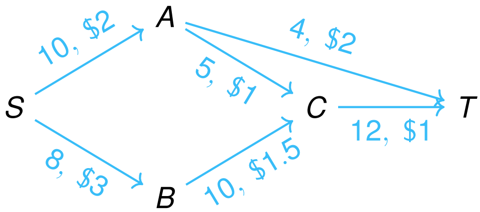

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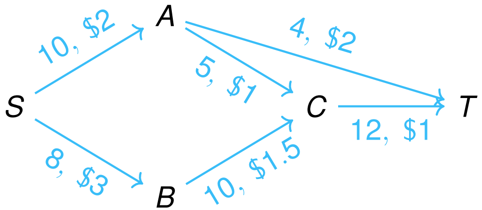
Dual Prediction:

- Shadow price: $\pi_{A \to C} \approx -1.5$
- Change in capacity: $\Delta = -5$.
- Predicted cost change:
  $\pi \cdot \Delta = (-1.5) \cdot (-5) = +7.5$ ✓

Economic Interpretation

The story of shadow prices in this network:



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- **Bottlenecks:** Arcs $A \to C$ and $A \to T$ are scarce and valuable
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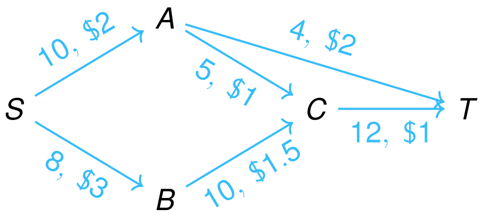

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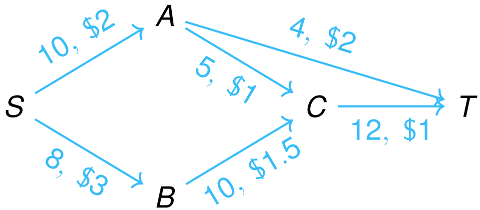

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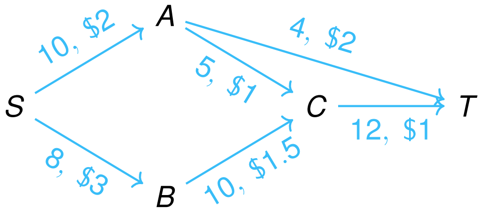

Investment Recommendation: If you can widen only one or two roads, pick $A \to C$ and $A \to T$.

Important Caveat on Shadow Prices: Only Valid Locally For Local “Small” Changes.

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That’s why we say: shadow prices are **locally valid**, not global.

When the Shadow Price Rule Fails

Example: Recall the Gridlock Gambit.

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Lesson: Shadow prices hold only **until the binding set changes.**

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Takeaway: Shadow prices are local. Once a new constraint becomes tight or slack, the slope changes.

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Tagline: Duals are not just math; they quantify *which constraints really matter*.