CS498: Algorithmic Engineering

Lecture 4: LP Relaxations, Vertex Cover, & When They Fail

Elfarouk Harb

University of Illinois Urbana-Champaign

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Outline

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- Use LPs to attack **discrete** problems on graphs and real-world settings.
- See how LPs give **approximate** solutions to NP-hard problems.
- See one case where LPs are **great** (Vertex Cover), and one case where they **fail badly** (Independent Set).

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Week 02 (this week): Linear Programming in depth

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Today: "What happens if we *pretend* the integer decisions are continuous?"

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  Today our discrete decisions will be $x \in \{0, 1\}$: turn on or off this variable.

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But we want a **smaller** cover.

Tiny Examples to Build Intuition

Path of length 3: $a - b - c$

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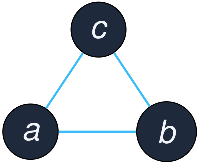

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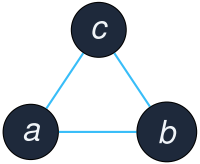
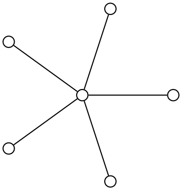


- Minimum vertex cover size?

Star: one center, many leaves



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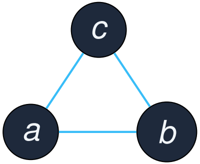

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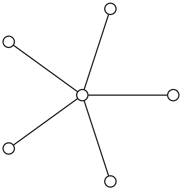

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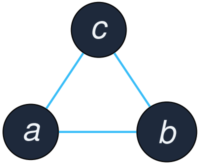

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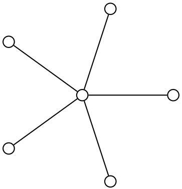

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Next step: turn this into an optimization model with variables.

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Still need constraints to enforce: *every edge is covered by at least one chosen endpoint.*

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Put everything together . . .

Integer Program for Minimum Vertex Cover

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This is our first full **integer program** in this course.

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$\Rightarrow$ Every vertex cover C corresponds to a feasible IP vector x.

Why Every Feasible IP Solution is a Vertex Cover

Now suppose we have a **feasible IP solution** x:

- $x_v \in \{0, 1\}$ for all v,
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$$\Rightarrow C \text{ is a vertex cover.}$$

Feasible IP solutions $\iff$ **vertex covers.**

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So **Integer Program** models **minimum vertex cover** exactly.

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Now this is a standard LP. Gurobi can solve it quickly.

Why LP$^\star \le$ OPT for Minimization

$[0,1]^n$

$$F_{\mathrm{IP}} \subseteq F_{\mathrm{LP}} \subseteq [0,1]^n$$

$$\mathrm{LP}^\star = \min_{x \in F_{\mathrm{LP}}} \sum_v x_v \;\le\; \min_{x \in F_{\mathrm{IP}}} \sum_v x_v = \mathrm{OPT}.$$

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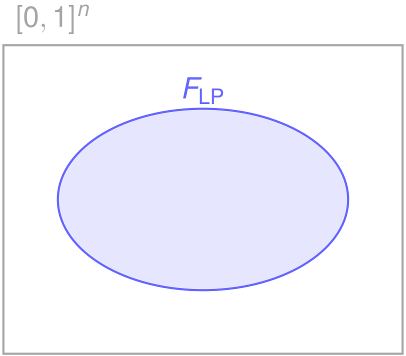

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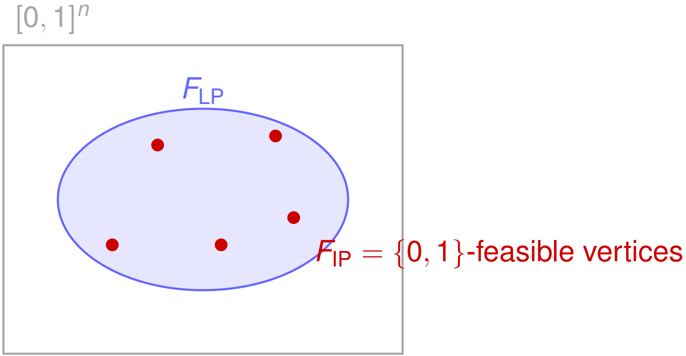

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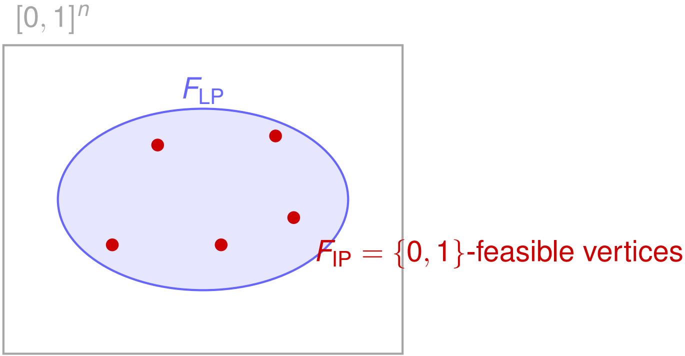

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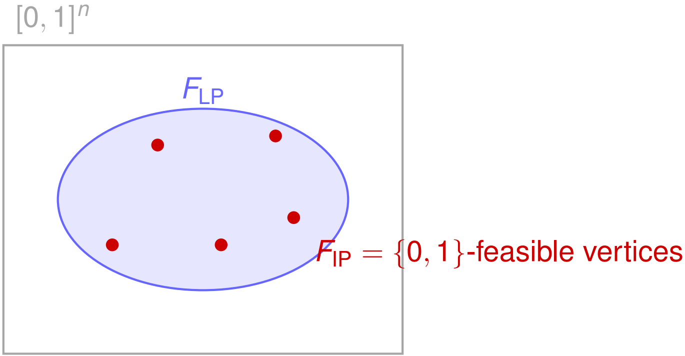

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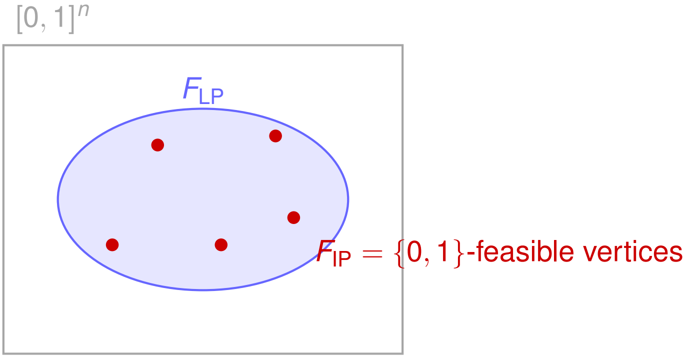

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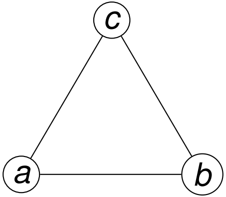

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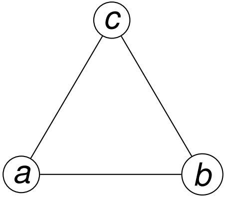

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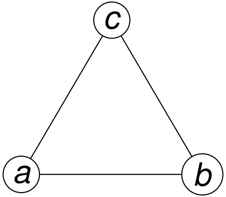

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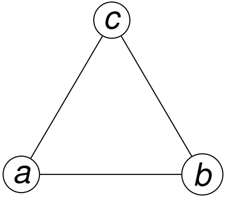

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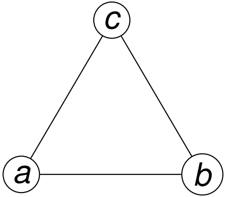

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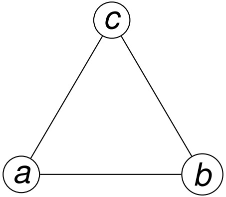

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Next: how to round $x^\star$ into an actual vertex cover C?

Rounding Scheme: The Half-Threshold Trick

Algorithm:

- Solve the LP and get optimal solution $x^\star$.

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1. Is C always a valid vertex cover?
2. How big can $|C|$ be compared to the true optimum vertex cover?

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$\Rightarrow$ C is always a **valid vertex cover**.

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Our rounded cover is at most a factor 2 worse than the best possible cover!

Gurobi Code Sketch for Vertex Cover LP

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from gurobipy import GRB

def solve_vertex_cover_lp(V, E):
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    # Minimize number of chosen vertices
    m.setObjective(gp.quicksum(x[v] for v in V), GRB.MINIMIZE)
    m.optimize()

    return m, x
```

Rounding the LP Solution in Code

```
def round_vertex_cover_lp(V, E):
    m, x = solve_vertex_cover_lp(V, E)
    C = {v for v in V if x[v].X >= 0.5}
    return C
```

Assignment Problem

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Goal: each driver does exactly one route; each route done by at most one driver; total cost is minimized.
This is a classic **assignment problem**. It can be written as an IP.

Assignment as IP (and LP)

Binary variables:

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LP relaxation: replace $x_{ij} \in \{0, 1\}$ by $0 \le x_{ij} \le 1$.

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- The constraint matrix is **totally unimodular**.
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Assignment problem is a case where LP relaxation is **perfect**. You can solve it exactly and integrally using an LP.

Gurobi Code Sketch for Assignment

```python
import gurobipy as gp
from gurobipy import GRB

def solve_assignment(drivers, routes, cost):
    m = gp.Model("assignment")
    m.Params.OutputFlag = 0

    # LP variables (we don't enforce integrality here!)
    x = m.addVars(drivers, routes, lb=0.0, ub=1.0, name="x")

    # each driver takes exactly one route
    for i in drivers:
        m.addConstr(gp.quicksum(x[i,j] for j in routes) == 1,  name=f"driver_{i}")

    # each route used at most once
    for j in routes:
        m.addConstr(gp.quicksum(x[i,j] for i in drivers) <= 1, name=f"route_{j}")

    m.setObjective(
        gp.quicksum(cost[i,j] * x[i,j] for i in drivers for j in routes), GRB.MINIMIZE
    )
    m.optimize()

    # inspect solution: you should see x[i,j] in {0,1}
    return m, x
```

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Next: an example where the LP relaxation is **terrible**.

Maximum Independent Set: Definition

Independent set $S \subseteq V$:

- No edge has both endpoints in S.
- Formally:

$$\forall (u, v) \in E : \text{ not } (u \in S \text{ and } v \in S).$$

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Find S with maximum size $|S|$ that is independent.

This is also NP-hard, and looks similar to Vertex Cover:

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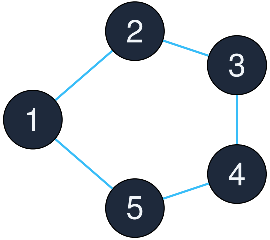
Now we ask the same question: **How good is this LP as an approximation?**

Example: 5-Cycle



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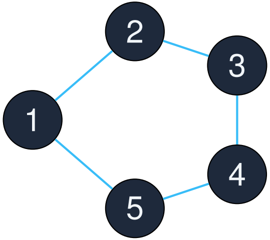

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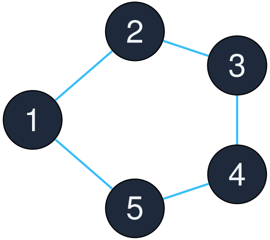

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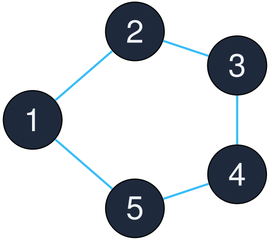

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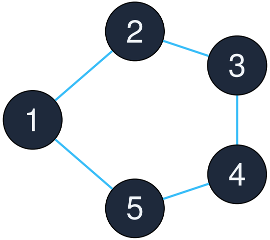

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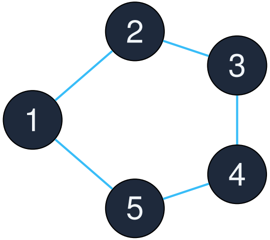

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$\text{LP}^\star \geq 2.5$ but $\text{OPT} = 2$. Not catastrophic yet.

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$\text{LP}^\star \geq n/2$ vs $\text{OPT} = 1$. **Gap factor** $\approx n/2$ **— terrible!**

Takeaways from MIS Example

- MIS IP and VC IP look very similar:
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LP relaxations are a tool, not magic, they can shine or fail.

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- Saw the **assignment problem** where LP relaxation is exact (integral polytope).
- Built the MIS IP and its LP relaxation, and demonstrated **huge integrality gaps** (clique example).

Summary: LP Relaxations, Successes, and Failures

What we did today:

- Built an **integer program** for Minimum Vertex Cover from scratch.
- Carefully showed the equivalence: vertex covers $\leftrightarrow$ feasible IP solutions.
- Relaxed integrality to get a Vertex Cover LP and proved:
  - $\mathrm{LP}^\star \leq \mathrm{OPT}$ (for minimization),
  - Simple threshold rounding gives a 2-approximation.
- Saw the **assignment problem** where LP relaxation is exact (integral polytope).
- Built the MIS IP and its LP relaxation, and demonstrated **huge integrality gaps** (clique example).

Next time: start doing Integer Programming *for real* (branch-and-bound, cutting planes, and more modeling tricks).