CS498: Algorithmic Engineering

Lecture 5: From LP Relaxations to Integer Solutions

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Week 03 – 02/03/2026

Outline

1. Setup and Motivation
2. From LPs to Integer Programs
3. Example: 0–1 Knapsack
4. Before Branch and Bound: The Naïve Approach
5. Branch and Bound: The Core Idea

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Where We Are in the Course

Week 02:

- Linear programming in depth – duality, sensitivity, LP as approximation.
- LP relaxations for Assignment (exact), Vertex Cover (2-approx) and Independent Set (huge gap).
- Assignment problem was special: LP = exact integer solution.

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This week:

- We want exact integer solutions, not approximations.
- We'll see how solvers *enforce integrality*.
- And why the strength of formulation decides speed.

Motivation: Relaxing vs Enforcing Integrality

Recall Vertex Cover LP:

$$\min \sum_{v} x_v \quad s.t.\ x_u + x_v \geq 1\ \forall (u, v) \in E,\ 0 \leq x_v \leq 1.$$

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Idea: Force $x_v \in \{0, 1\} \implies$ an **Integer Program**.
LP = “relax your morals.”
IP = “follow the rules exactly.”

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Integer Linear Programs (Definition)

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Special cases:

- All x_i continuous or $I = \emptyset \rightarrow$ LP.
- $x_i \in \{0, 1\}$ for $i \in I \rightarrow$ **Binary Integer Linear Program**.

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Usually we only need 0/1 decisions; other integers can be counted in bits.

Representing General Integers with Binaries

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Example 2: Large integer range

$$x \in \{0, \dots, 20\} \quad \Rightarrow \quad x = y_1 + 2y_2 + 4y_3 + 8y_4 + 16y_5, \quad y_i \in \{0, 1\} \text{ and } x \leq 20.$$

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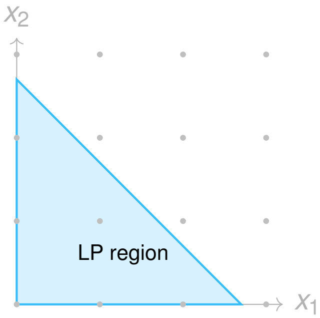
Key point: Solvers enforce integrality through binary representations, so focusing on **binary programs** is sufficient in theory and practice.

Geometry of IPs



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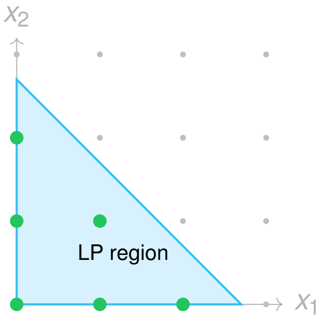

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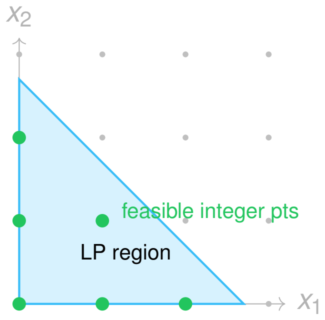

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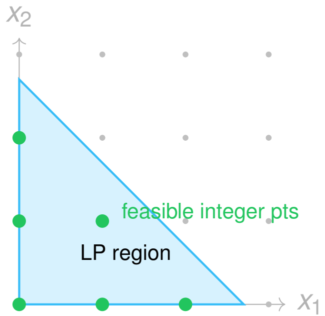

Key idea:

- LP feasible set = convex polytope.
- IP feasible set = subset of lattice points.

Relaxation Relationship

Let F_{IP} be feasible region for IP. Let F_{LP} be feasible region for LP.

For a minimization problem:

$$\text{LP}^* = \min_{x \in F_{LP}} c^\top x \leq \min_{x \in F_{IP}} c^\top x = \text{IP}^*.$$

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Meaning: LP solution is an **optimistic bound** on what's achievable with integers. Sometimes bound = exact (answer is already integral, like assignment problem). Sometimes it's not $\rightarrow$ we need to *search.*

A Tiny Illustrative IP

$$\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & 2x_1 + 3x_2 \geq 5, \\
& 0 \leq x_i \leq 3, \quad i = 1, 2, \\
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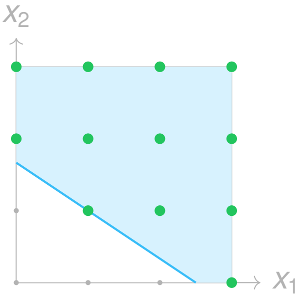

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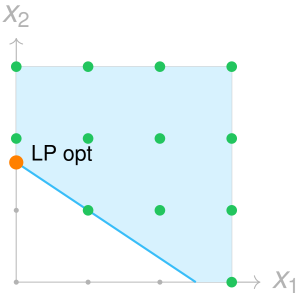

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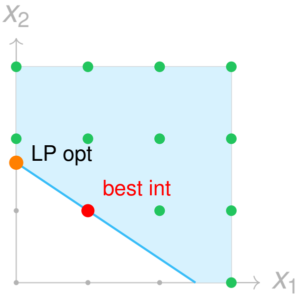

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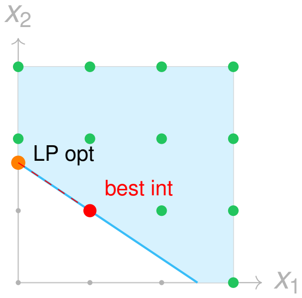
Additive gap $= 2 - \frac{5}{3} = \frac{1}{3} \approx 0.33 \Rightarrow$ we'll use this gap as a *bound*.



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Which items should we pack to maximize value without exceeding capacity?

Variables and Model (1/2)

Define binary decision variables:

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Objective:

$$\max\ 2x_1 + 2x_2 + 5x_3 + 6x_4.$$

Bruteforce Attempt (2/2)

$$\begin{aligned}
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$$\text{s.t. } x_1 + 2x_2 + 2x_3 + 3x_4 \leq 4,$$
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Feasible integer solutions (respect weight ≤ 4):

(x_1, x_2, x_3, x_4)	Weight	Value
$(0,0,0,0)$	0	0
$(0,0,0,1)$	3	6
$(0,0,1,0)$	2	5
$(0,1,0,0)$	2	2
$(0,1,1,0)$	4	7
$(1,0,0,0)$	1	2
$(1,0,0,1)$	4	8
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Examples of infeasible (too heavy) combinations:

(x_1, x_2, x_3, x_4)	Weight	Value
$(0, 0, 1, 1)$	5	11 (too heavy)
$(0, 1, 0, 1)$	5	8 (too heavy)
$(1, 1, 1, 1)$	8	15 (too heavy)

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Best feasible integer solution:

$$(x_1, x_2, x_3, x_4) = (1, 0, 0, 1) \quad \Rightarrow \quad \text{value } = 8.$$

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Pseudo-code:

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for x in itertools.product([0,1], repeat=k):
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For k binary variables, that's 2^k possibilities.
Works fine for $k \leq 20 \ldots$ catastrophic after that.
Exponential time: even $k = 50$ gives 1.1×10^{15} cases.

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So B&B = "smart brute force guided by LPs."

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LP provides an optimistic upper bound of 9: “You can’t do better than 9!”

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This is where **Branch and Bound (B&B)** enters.

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6. Prune branches that are infeasible or worse than best integer so far.

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where z^*_{node} is the unknown optimal integer value in that subtree.

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At the root, our goal is to shrink the gap $UB_{\text{root}} - LB$ until $UB_{\text{root}} = LB$, proving optimality.

Knapsack Problem at Root

Consider the 0–1 knapsack:

$$\max\ 2x_1 + 2x_2 + 5x_3 + 6x_4$$

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LP relaxation (allow $0 \le x_i \le 1$).

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = \tfrac{1}{3},$$

$$z_{\text{LP}} = 2 \cdot 1 + 2 \cdot 0 + 5 \cdot 1 + 6 \cdot \tfrac{1}{3} = 2 + 0 + 5 + 2 = 9.$$

So initially:

$$UB_{\text{root}} = 9, \quad LB = -\infty \text{ (no incumbent yet).}$$

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Fractional variable: $x_4 = \frac{1}{3} \Rightarrow$ branch on x_4.

First Branch: Fixing x_4

Branch on the fractional variable x_4:

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We will explore:

- The left subtree (Node A, $x_4 = 1$) first.
- Then the right subtree (Node B, $x_4 = 0$).

Left Subtree: Node A ($x_4 = 1$)

At Node A, we fix $x_4 = 1$. The constraint becomes:

$$x_1 + 2x_2 + 2x_3 + 3 \cdot 1 \leq 4 \;\Rightarrow\; x_1 + 2x_2 + 2x_3 \leq 1.$$

The LP relaxation at Node A is:

$$\max \; 2x_1 + 2x_2 + 5x_3 + 6 \cdot 1 = 6 + 2x_1 + 2x_2 + 5x_3$$

$$\text{s.t. } x_1 + 2x_2 + 2x_3 \leq 1, \quad 0 \leq x_1, x_2, x_3 \leq 1.$$

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$$x_1 = 0,\; x_2 = 0,\; x_3 = \tfrac{1}{2},\; x_4 = 1,$$
$$z_{\text{LP}} = 6 + 5 \cdot \tfrac{1}{2} = 6 + 2.5 = 8.5.$$

So

$$UB_A = 8.5.$$

The solution is fractional in $x_3 \Rightarrow$ branch on x_3.

Left Subtree: Children of Node A ($x_4 = 1$)

Branch on x_3 at Node A:

$$\text{Node A1: } x_4 = 1,\ x_3 = 1, \qquad \text{Node A2: } x_4 = 1,\ x_3 = 0.$$

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$$x_1 + 2x_2 + 2 \cdot 1 + 3 \cdot 1 \leq 4 \ \Rightarrow\ x_1 + 2x_2 + 5 \leq 4 \ \Rightarrow\ x_1 + 2x_2 \leq -1,$$

which is impossible. So Node A1 is **infeasible** and thus **fathomed**.

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Node A2: $x_4 = 1,\ x_3 = 0.$ LP relaxation at Node A2:

$$\max\ 6 + 2x_1 + 2x_2 \quad \text{s.t. } x_1 + 2x_2 \leq 1,\ 0 \leq x_1, x_2 \leq 1.$$

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The LP solution is $x_2 = 0,\ x_1 = 1$:

$$x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 1, \quad z = 6 + 2 = 8.$$

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$$x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 1, \quad z = 6 + 2 = 8.$$

This is an **integer** solution, so we update the incumbent

$$LB = 8.$$

Summary of Left Subtree (Node A, $x_4 = 1$)

In the left subtree ($x_4 = 1$):

- Node A1: infeasible $\Rightarrow UB_{\text{A1}} = -\infty$.
- Node A2: LP solution is integer solution

$$(x_1, x_2, x_3, x_4) = (1, 0, 0, 1), \quad z = 8.$$

So $UB_{\text{A2}} = 8$ and update global incumbent to $LB = 8$.

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Thus, for Node A overall:

$$UB_A = \max\{UB_{\text{A1}}, UB_{\text{A2}}\} = \max\{-\infty, 8\} = 8,$$

We now turn to the right subtree, Node B ($x_4 = 0$), starting from:

$$UB_{\text{root}} = 9, \quad LB = 8.$$

Right Subtree: Node B ($x_4 = 0$)

At Node B, we fix $x_4 = 0$. The constraint simplifies to:

$$x_1 + 2x_2 + 2x_3 + 3 \cdot 0 \leq 4 \;\Rightarrow\; x_1 + 2x_2 + 2x_3 \leq 4.$$

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So $UB_B = 8$. However, our incumbent is already $LB = 8$. Therefore $UB_B = 8 \leq LB = 8$.

Conclusion: Even though the LP at Node B is fractional, *no integer solution in this subtree can improve the incumbent*. We **fathom Node B by bound**, without branching further.

Final Summary of the B&B Tree

We have explored all necessary branches:

- Left subtree (Node A, $x_4 = 1$):
  - Node A1, $x_3 = 1, x_4 = 1$: infeasible (fathomed by infeasibility).
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    This solution becomes the incumbent: $LB = 8$. No branching.

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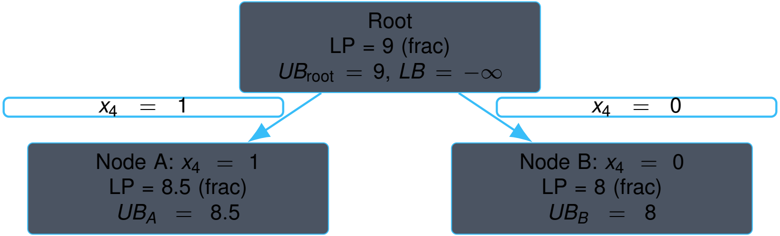
- Update $UB_{\text{root}} = \max(UB_A, UB_B) = \max(8, 8) = 8 = LB$. Terminate!

Best integer value found:

$$z^* = LB = 8, \quad x^* = (1, 0, 0, 1).$$

Root bounds at termination: $UB_{\text{root}} = 8, \quad LB = 8$. Since $UB_{\text{root}} = LB$, the

Evolving Tree (Stage 1)



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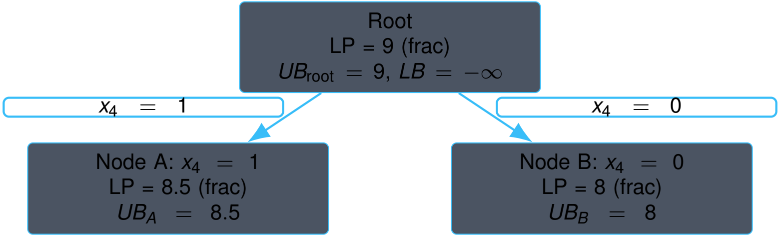
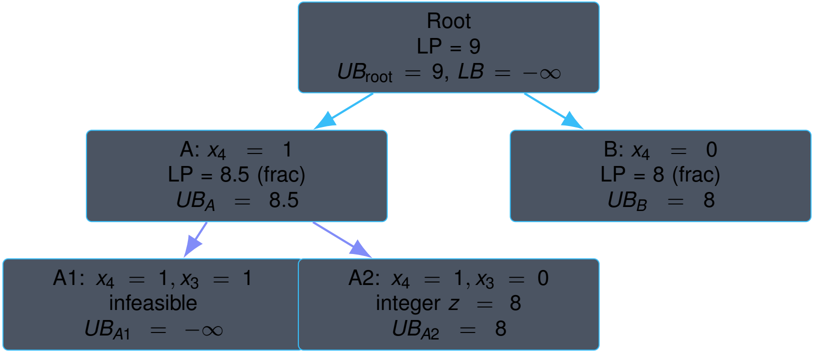


Both children are fractional.

We next expand Node A.

Evolving Tree (Stage 2a)

Expand Node A (fractional in x_3):



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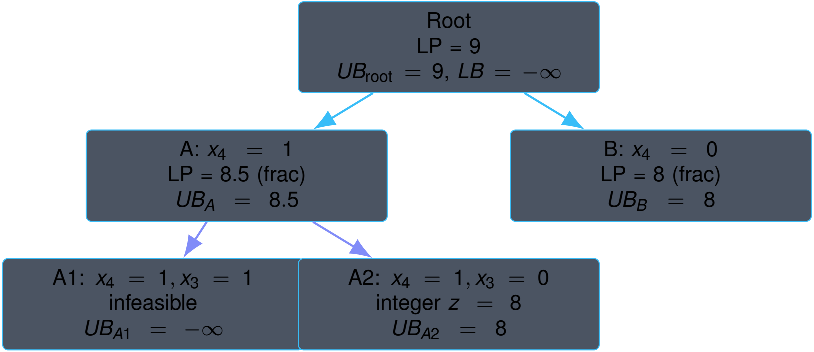

Node A1 is infeasible $\Rightarrow$ it contributes no feasible integer solution.

Node A2 has an **integer** LP solution with value $z = 8$.

Evolving Tree (Stage 2b)

Using the information from A1 and A2, we update bounds at Node A and the root.

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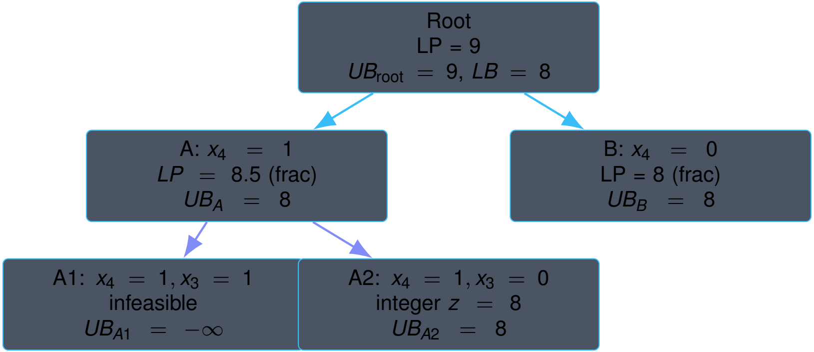

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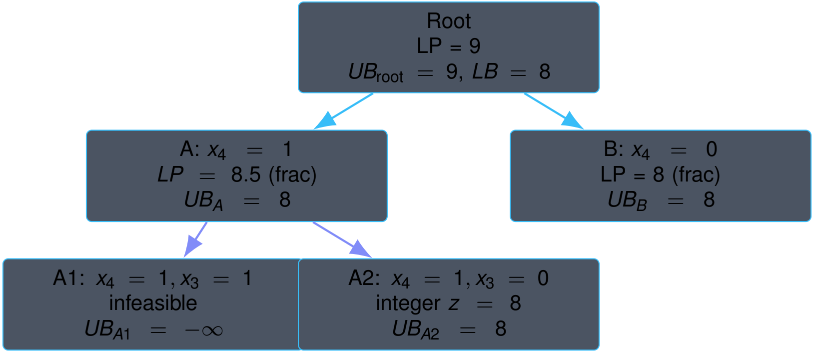
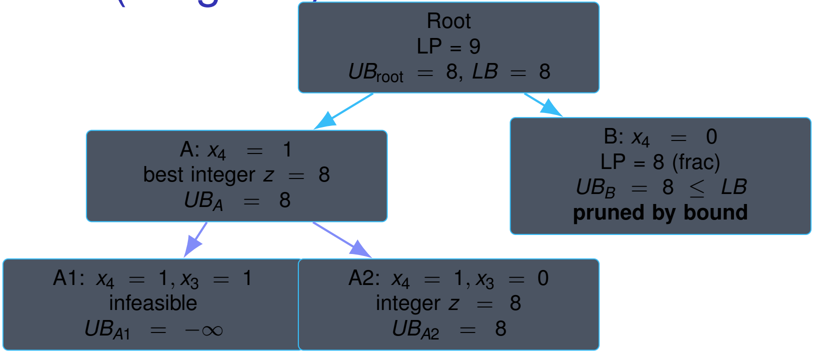


From A2, we obtain an incumbent with value $z = 8 \Rightarrow$ update global $\mathbf{LB} = 8$.
Node A's subtree cannot contain any solution better than 8, so $UB_A = 8$.
Node B remains active and will be considered next.

Evolving Tree (Stage 2c)



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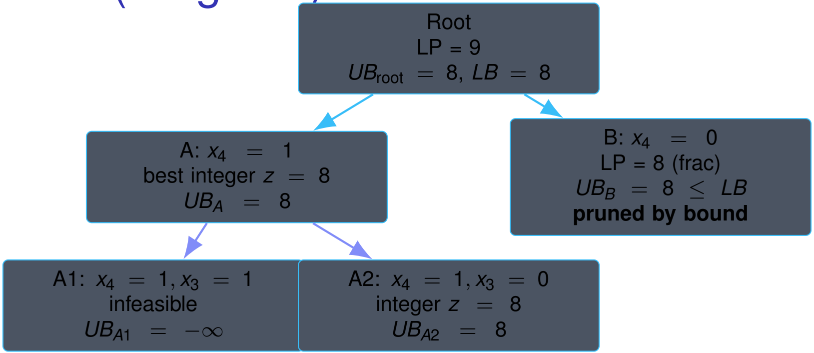

Since $UB_B = 8 \leq LB = 8$, Node B cannot contain any better integer solution.
Node B is **fathomed by bound**.
Update $UB_{\text{root}} = \max(UB_A, UB_B) = \max(8, 8) = 8$.
All nodes are now resolved, and $UB_{\text{root}} = LB = 8$, proving optimality of the incumbent $x^* = (1, 0, 0, 1)$ with value 8.

Bounding and Fathoming Summary

Fathoming rules (for maximization)

A node can be skipped (“fathomed”) if:

- LP is infeasible (e.g. Node A1), or
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Each LP bound tightens the global UB.

Algorithmic Behavior

- Each LP call provides a bound.
- Each integer solution provides an incumbent.
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Visualize B&B as a dialogue:

LP: "At best, I can get 11in this node"   LB: "I already have 12!"

The gap tells us how much hope remains.

Node Selection Strategies

The order that we process nodes in B&B matters. We did the order root $\rightarrow A \rightarrow A1 \rightarrow A2 \rightarrow B$. **Two classical options:**

- **Depth-First Search (DFS):** quickly finds feasible integers (good for LB).
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Modern solvers use hybrids: DFS early $\to$ Best-Bound once good incumbent found.

Branching Variable Choice

If there is only one variable that is fractional, just branch on it. But what if there is more than one?
Which variable to branch on?

- Most fractional ($x_i \approx 0.5$) $\to$ balances search.
- Greatest effect on objective (pseudo-costs).
- Domain-specific heuristics (e.g., branching on vertex degree).

Also, which branch to explore first? (i.e. $x_3 = 0$ or $x_3 = 1$ first?).

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Smart branching = smaller tree = faster solve. Tons of heuristics.

Summary of B&B Workflow

1. Solve LP relaxation.
2. If integral, update incumbent. If fractional $\rightarrow$ branch.
3. Update bounds (both UB and LB of all nodes).
4. Prune by infeasibility or domination (fathoming).
5. Repeat until UB = LB.

Gurobi Example: Integer Knapsack

```python
import gurobipy as gp
from gurobipy import GRB

values  = [10, 7, 4]
weights = [5, 4, 3]
W = 7

m = gp.Model("knapsack_ip")
x = m.addVars(3, vtype=GRB.BINARY, name="x") #Only new thing

m.addConstr(sum(weights[i]*x[i] for i in range(3)) <= W)
m.setObjective(sum(values[i]*x[i] for i in range(3)), GRB.MAXIMIZE)
m.optimize()
```

Extracting Solver Statistics

After `m.optimize()`:

```
print("Optimal value:", m.ObjVal)
print("Nodes explored:", m.NodeCount)
print("Best bound:", m.ObjBound)
print("Gap:", m.MIPGap)
for v in x.values():
    print(v.VarName, v.X)
```