CS498: Algorithmic Engineering

Lecture 8: Row Generation (LP), Minimum Cut, and The Ellipsoid Method

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Outline

1. Quick Note from Last Lecture
2. Min *s*–*t* Cut Warm-up
3. Row Generation for Linear Programs
4. Case Study (LP): Minimum *s*–*t* Cut

Practical MINLP in Gurobi

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Engineering Note: Just like Big-M, Spatial B&B relies heavily on **variable bounds** (L_x, U_x) to build tight envelopes. Always bound your continuous variables in MINLP!

Practical MINLP in Gurobi

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```
m = gp.Model("bilinear_example")
x = m.addVar(lb=0, ub=10, name="x")  # Bounds are CRITICAL for envelopes!
y = m.addVar(lb=0, ub=10, name="y")
z = m.addVar(name="z")

# 1. Add the bilinear constraint directly
#    Gurobi detects this is non-convex
m.addConstr(z == x * y)

# 2. REQUIRED: Enable non-convex handling
#    0 = error if non-convex (default)
#    2 = translate to McCormick & use Spatial B&B
m.setParam("NonConvex", 2)

m.optimize()
```

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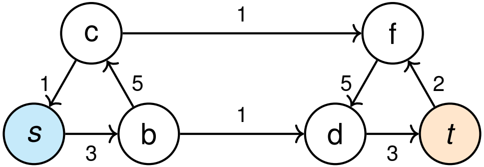

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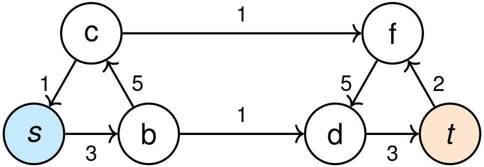

Intuition: cutting the weak links disconnects s from t.

Definition: s–t Cut

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Mathematical Definition (Partition): An s–t cut is a partition $(S, V \setminus S)$ such that $s \in S$ and $t \notin S$. The capacity of the cut is the weight of edges **leaving** S:

$$w(\delta^+(S)) = \sum_{\substack{(u,v)\in E \\ u\in S,\ v\notin S}} w_{uv}$$

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Why this definition?

Why do we focus on partitions? Because any **minimal** set of edges that separates s from t forms a cut $\delta^+(S)$, where S is the set of nodes reachable from s.

The Max-Flow Min-Cut Theorem [CS473]

How do we compute the Minimum Cut?

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- **Flow** is naturally a Linear Programming (LP) problem (continuous variables).
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This theorem allows us to solve the discrete Cut problem efficiently using Linear Programming and Max-Flow Algorithms!

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Caution

The formulation we show next is useful for understanding row-generation, but it is **not** the standard textbook flow formulation you might see in other contexts. For standard flow algorithms, see CS473.

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So the minimum s–t cut is defined by the set:

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And the cut value is $w(\delta^+(S)) = 2$.

In Python: NetworkX Minimum s–t Cut

```
import networkx as nx

G = nx.DiGraph() # Explicitly Directed

# Add edges with capacities
G.add_edge("s","b", capacity=3)
G.add_edge("b","c", capacity=2)
G.add_edge("c","s", capacity=2)

G.add_edge("d","t", capacity=3)
G.add_edge("t","f", capacity=2)
G.add_edge("f","d", capacity=2)

G.add_edge("b","d", capacity=1)
G.add_edge("c","f", capacity=1)

# NetworkX uses Max-Flow Min-Cut theorem internally
cut_value, partition = nx.minimum_cut(G, "s", "t")
S, T = partition

print("min st cut value =", cut_value)
print("S =", S)
```

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Row Generation: When Constraints Are Too Many to Write

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Key idea

Most constraints are *not* binding at optimum.
So: start small, then add only what you need.

The Abstraction: Full LP vs Restricted LP

Let the true LP be L:

$$\min c^\top x \quad \text{s.t. } a_k^\top x \le b_k \ \ \forall k \in \mathcal{K}, \quad x \in X$$

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- x^* violates none $\Rightarrow$ x^* is optimal for L, we are done!
- x^* violates some constraint in $L \Rightarrow$ add it to L'.

Sometimes 1 Constraint is Already Enough

Toy LP with many constraints (conceptually):

$$\min\ x \qquad \text{s.t. } x \geq 1,\ x \geq 2,\ x \geq 3,\ \ldots,\ x \geq 10^{100}.$$

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Outcome (we are done)

No violated constraints $\Rightarrow x^*$ is optimal for the full LP L.

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Outcome

Some violated constraints, for example $x \geq 2$.

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Row generation = repeatedly call the oracle and add returned rows.

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   - Else: stop. Current x^* is optimal for full LP.

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So the lecture question becomes:

Can this actually happen? Can we have a huge constraint set, but a fast separation oracle?

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Now we model this as an optimization problem.

Step 1: What Are Our Decisions?

For every edge $e \in E$, define:

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If we cut edge e, we pay cost w_e.

Step 2: Objective Function

We want to minimize total cutting cost:

$$\min \sum_{e \in E} w_e \, y_e$$

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But we have not yet enforced that s and t are disconnected.

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Every path from s to t contains at least one cut edge.

In other words:

$$\text{For every } s\text{–}t \text{ path } P, \quad \sum_{e \in P} y_e \geq 1.$$

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That means the path still exists.
So to eliminate the path, we must enforce:

$$\sum_{e \in P} y_e \geq 1.$$

Full Integer Programming Model For Min $s - t$ Cut.

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\begin{aligned}
\min \quad & \sum_{e \in E} w_e y_e \\
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This is a correct formulation of minimum s–t cut.

Now Look Carefully at the Constraints

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How many s–t paths are there?

Problem

Potentially exponentially many.

Relax Integrality: Full Linear Programming Model

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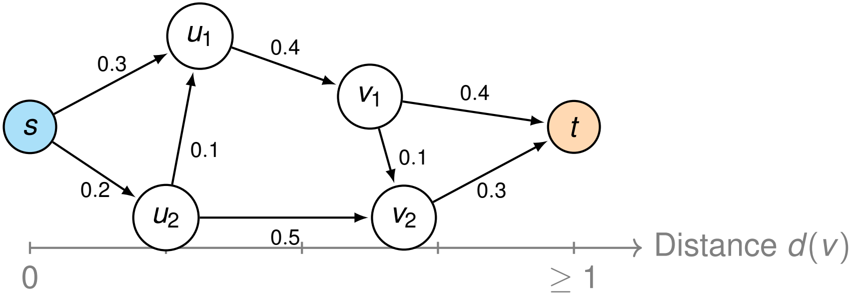
Then:

$$d(s) = 0, \qquad d(t) \geq 1.$$

Visualizing the Distance Metric



We arrange the nodes on the horizontal axis according to their shortest path distance $d(v)$ from s using the fractional weights y_e.

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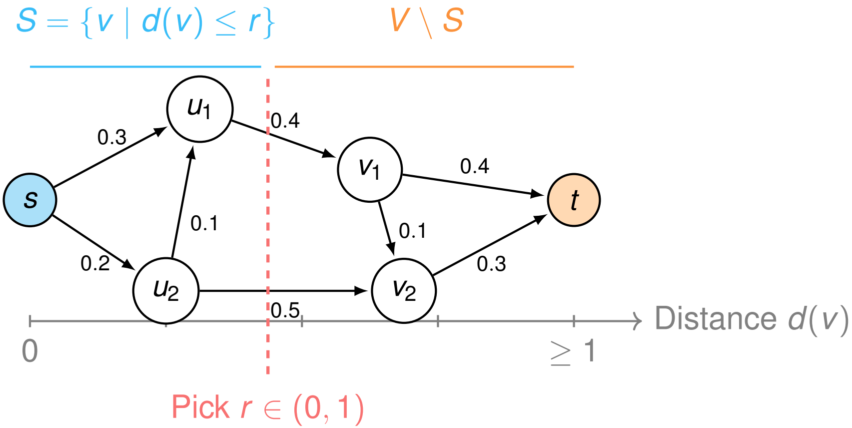
This is an s–t cut since:

$$d(s) = 0 \leq r, \qquad d(t) \geq 1 > r.$$

The Random Threshold Cut



Algorithm: Pick a radius $r \in (0, 1)$. Say $r = 0.35$
Cut all edges (u, v) where u is "inside" ($d(u) \leq r$) and v is "outside" ($d(v) > r$).

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Swap summation and integration:

$$\int_0^1 \mathrm{cost}(C(r))\, dr = \int_0^1 \sum_{e \in C(r)} w_e\, dr = \sum_{e=(u,v) \in E} w_e \cdot \Big(\text{length of } r \text{ where } e \text{ is in } C(r)\Big).$$

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$$\int_0^1 \mathrm{cost}(C(r))\, dr.$$

The cost of a cut is:

$$\mathrm{cost}(C(r)) = \sum_{e \in C(r)} w_e.$$

Swap summation and integration:

$$\int_0^1 \mathrm{cost}(C(r))\, dr = \int_0^1 \sum_{e \in C(r)} w_e\, dr = \sum_{e=(u,v) \in E} w_e \cdot \Big(\text{length of } r \text{ where } e \text{ is in } C(r)\Big).$$

An edge (u, v) is cut exactly when $d(u) \le r < d(v)$. The length of this interval is $\max\{0,\, d(v) - d(u)\}$.

Triangle Inequality

Shortest-path distances satisfy (Triangle inequality):

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Summing over all edges:

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This equals the LP cost.

Conclusion: A Good Integral Cut Exists

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$$\int_0^1 \mathrm{cost}(C(r))\, dr \le \mathrm{LP\ Cost}.$$

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We proved that any fractional y is an average of integer cuts. A vertex cannot be an average of other things. Therefore, if y is a vertex, it must inherently be an integer cut itself.

Full LP Model For Min $s - t$ Cut.

$$
\begin{aligned}
\min \; & \sum_{e \in E} w_e y_e \\
\text{s.t.} \; & \sum_{e \in P} y_e \geq 1 \quad \forall \; s\text{–}t \text{ paths } P \\
& 0 \leq y_e \leq 1
\end{aligned}
$$

This is an integral polytope. Solving it gives us $y_e \in \{0, 1\}$.

Now the Real Question

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How can we solve it?

Row Generation Framework

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Given current solution y, we want to see if any constraint is violated.

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If shortest path length < 1, that path gives a violated constraint.

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2. Solve LP.
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5. Repeat.

This is real LP row generation.

Step 1: Build the Restricted LP

```
m = gp.Model("min_st_cut_path_LP")

# y_e in [0,1]
y = m.addVars(E, lb=0.0, ub=1.0,
              vtype=GRB.CONTINUOUS, name="y")

# Objective: minimize total cut capacity
m.setObjective(
    gp.quicksum(c[e] * y[e] for e in E),
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At this point:

- No path constraints yet.
- LP will initially set all $y_e = 0$.

Step 2: Row Generation Loop

```python
MAX_ITERS = 100

for it in range(MAX_ITERS):
    m.optimize()
    ystar = {e: y[e].X for e in E}

    # Build graph with edge weights = y*
    G = nx.DiGraph()
    for (u,v) in E:
        G.add_edge(u, v, weight=ystar[(u,v)])

    dist, path = nx.single_source_dijkstra(G, s, t)
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Now we check whether the shortest path violates a constraint.

Step 3: Check Violation

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if dist >= 1:
    print("No violated path constraints.")
    break
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All path constraints are satisfied.

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```

If shortest path length ≥ 1:

Termination

All path constraints are satisfied.

We are done.

Step 4: Add Violated Constraint

```python
# Add constraint for violated path
m.addConstr(
    gp.quicksum(
        y[(path[i], path[i+1])]
        for i in range(len(path)-1)
    ) >= 1
)
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Then we re-solve the LP.

Why This Works

- The LP has exponentially many path constraints.

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- Separation Oracle reduces to shortest path (poly-time Dijkstra).
- Therefore, row generation solves the LP.

And because min s–t cut LP is integral, the final solution is actually a true cut.

Final Min *s*-*t* Code

```
import gurobipy as gp
import networkx as nx
s, t = "s", "t"
E = [("s","a"),("s","b"),("a","t"),("b","t"),("a","b")]
c = {("s","a"): 2, ("s","b"): 1, ("a","t"): 1, ("b","t"): 3, ("a","b"): 1}
m = gp.Model("min_st_cut_path_LP")
y = m.addVars(E, lb=0.0, ub=1.0, vtype=gp.GRB.CONTINUOUS, name="y")
m.setObjective(gp.quicksum(c[e] * y[e] for e in E), gp.GRB.MINIMIZE)

for it in range(100):
    m.optimize()
    # Extract current solution
    ystar = {e: y[e].X for e in E}
    # Build graph with edge weights = ystar
    G = nx.DiGraph()
    for (u,v) in E: G.add_edge(u, v, weight=ystar[(u,v)])
    # Shortest path from s to t
    dist, path = nx.single_source_dijkstra(G, s, t)
    # Check violation
    if dist >= 1: break
    m.addConstr(gp.quicksum(y[(path[i], path[i+1])]  for i in range(len(path)-1)) >= 1)
#{('s', 'a'): 0.0, ('s', 'b'): 1.0, ('a', 't'): 1.0, ('b', 't'): 0.0, ('a', 'b'): 1.0}
```

Efficiency: Cold Start vs. Hot Start

We are solving a sequence of Linear Programs: $LP_0, LP_1, LP_2 \ldots$ where each is slightly larger than the last.

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- We throw away the previous optimal basis.
- The solver must start from scratch every time.
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The Smart Approach (Hot Start):

- Keep the `gp.Model` object alive.
- Just add the new constraint to the existing object.
- **Benefit:** Gurobi re-uses the internal state (basis) from the last solve.

Implementation: Where to put the Model?

Bad (Cold Start)

```
# Re-creating the model
# inside the loop is slow!

for it in range(100):

    m = gp.Model()  # <--- BAD
    y = m.addVars(...)

    # You have to re-add
    # ALL previous constraints
    # manually here.

    m.optimize()
```

Good (Hot Start)

```
# Create model ONCE
m = gp.Model()      # <--- GOOD
y = m.addVars(...)

for it in range(100):

    # Solve existing model
    m.optimize()

    # ... find violation ...

    # Modifies existing matrix
    m.addConstr(...)
```

In the **Good** case, ‘m.optimize()‘ detects the existing basis and performs a ”Warm Start” or ”Hot Start” automatically.

Under the Hood: Dual Simplex

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When we call ‘m.optimize()‘ after adding a constraint, Gurobi typically switches to the **Dual Simplex** algorithm.

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Algorithm Choice

When we call ‘m.optimize()‘ after adding a constraint, Gurobi typically switches to the **Dual Simplex** algorithm.

It starts from the previous basis and usually requires only a few pivots to fix the infeasibility, rather than traversing the whole polytope again.

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If a polytope admits a polynomial-time separation oracle for its constraints, then we can optimize a linear function over it (LP) in polynomial time (ellipsoid method).

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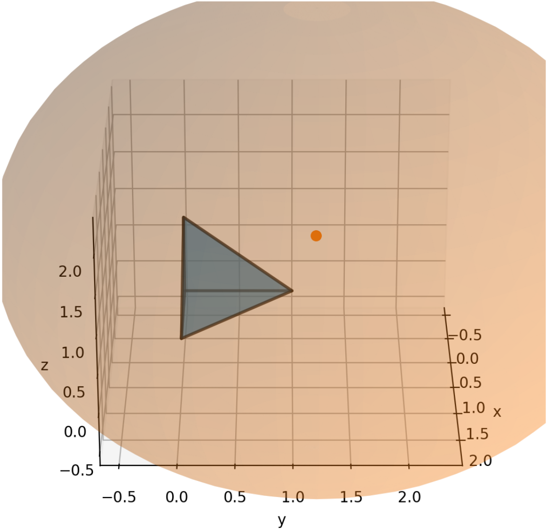
So for LPs:

- exponential constraints are not automatically hopeless,
- if we can separate efficiently.

Ellipsoid Method (3D) — iter 0 (show ellipsoid)
infeasible center | constraint cut | max-axis≈2.800



Ellipsoid Method (3D) — iter 0 (show cut plane)
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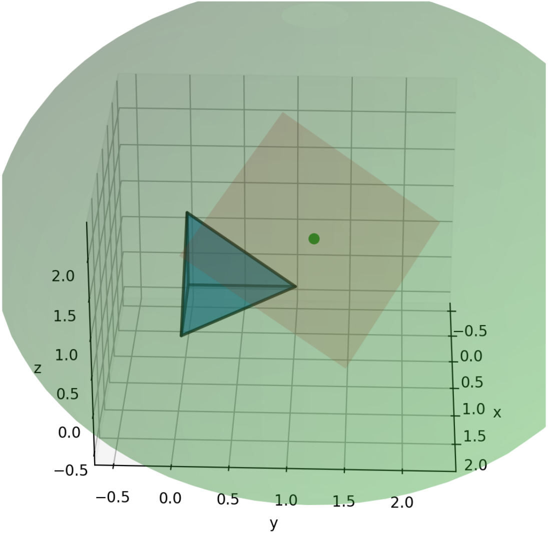
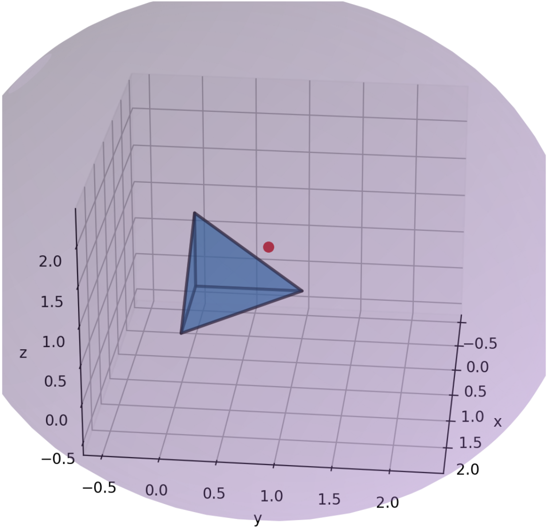
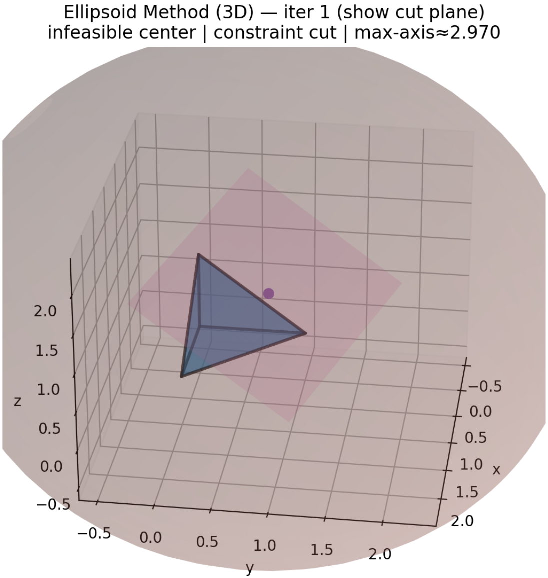


Ellipsoid Method (3D) — iter 1 (show ellipsoid)
infeasible center | constraint cut | max-axis≈2.970

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z
2.0
1.5
1.0
0.5
0.0
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y
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0.5
1.0
1.5
2.0
x
−0.5
0.0
0.5
1.0
1.5
2.0

Ellipsoid Method (3D) — iter 2 (show ellipsoid)
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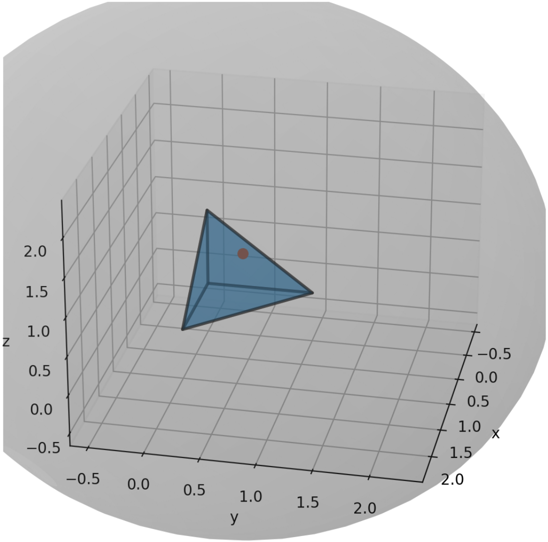

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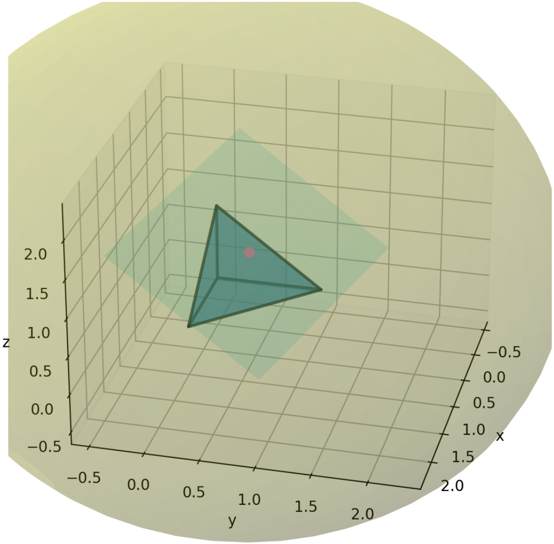
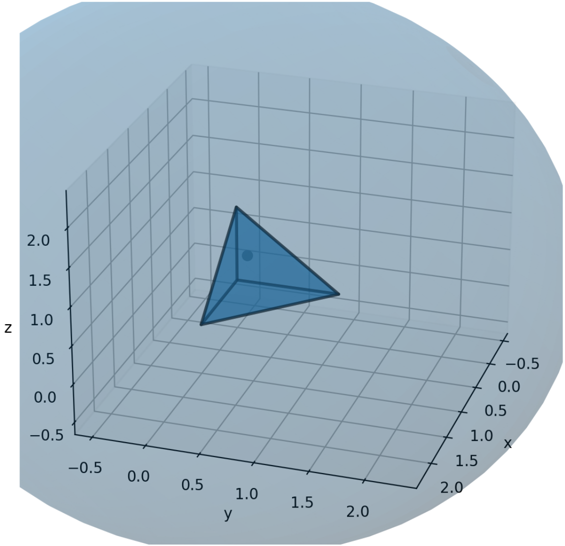


Ellipsoid Method (3D) — iter 3 (show ellipsoid)
infeasible center | constraint cut | max-axis≈3.341



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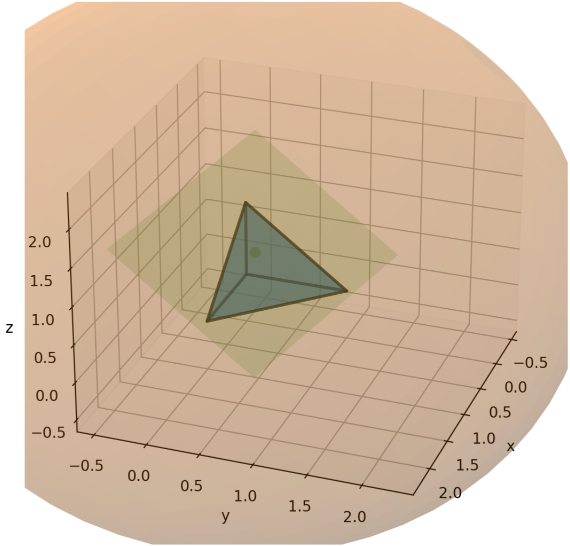
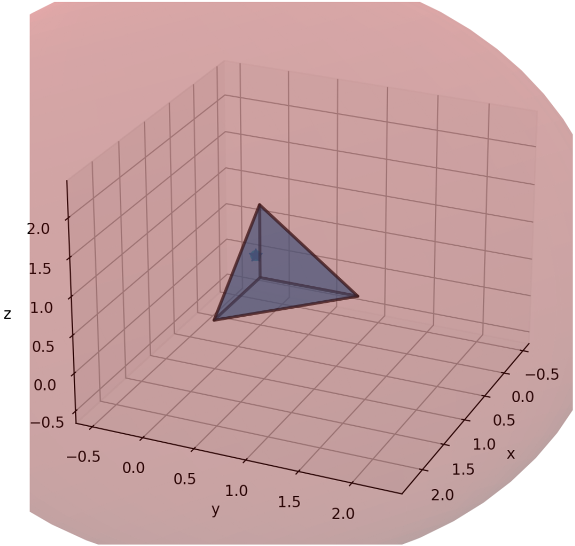


Ellipsoid Method (3D) — iter 4 (show ellipsoid)
feasible center | objective cut | max-axis≈3.544 | best d^T x≈0.965



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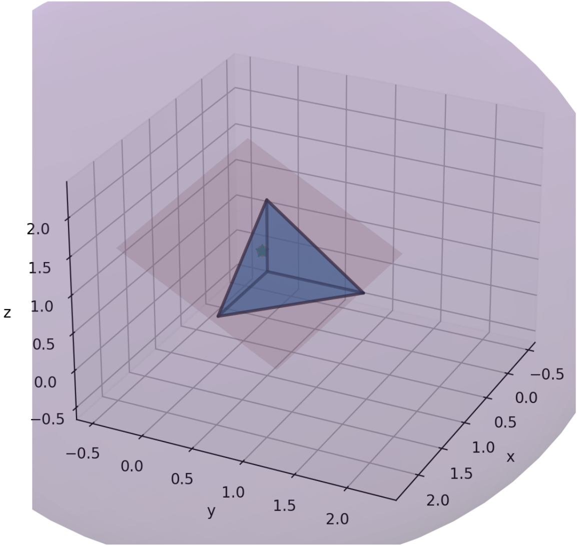
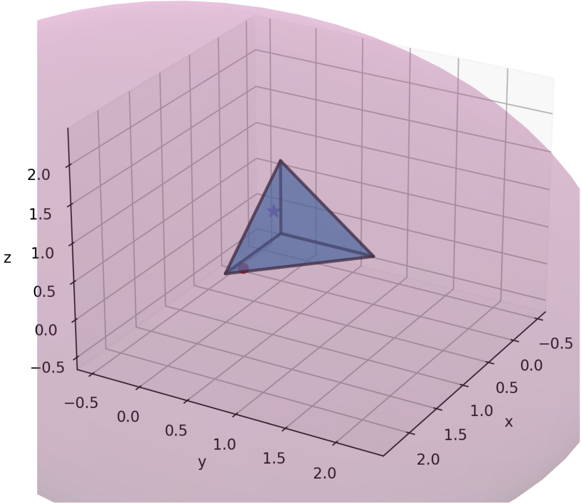


Ellipsoid Method (3D) — iter 5 (show ellipsoid)
infeasible center | constraint cut | max-axis≈3.759 | best d^T x≈0.965



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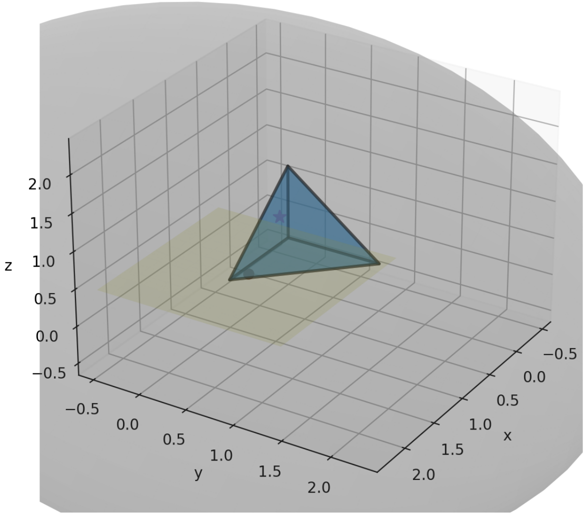

Ellipsoid Method (3D) — iter 6 (show ellipsoid)
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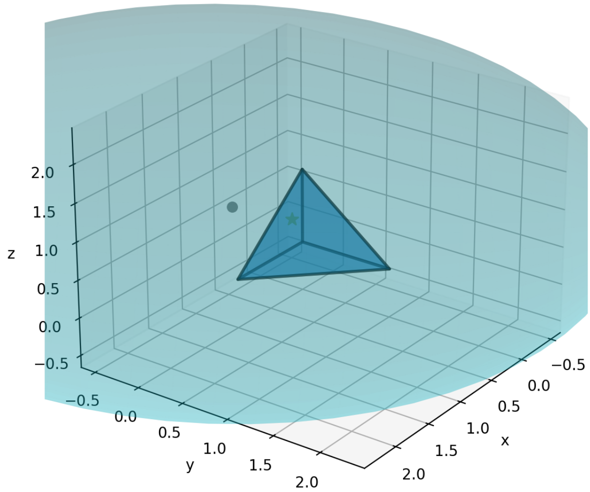

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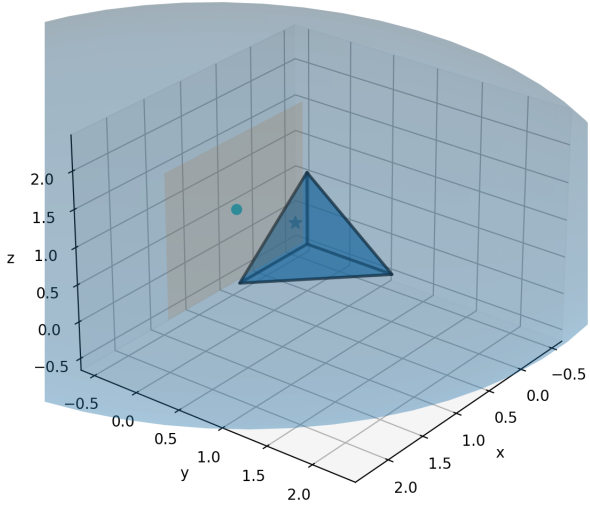
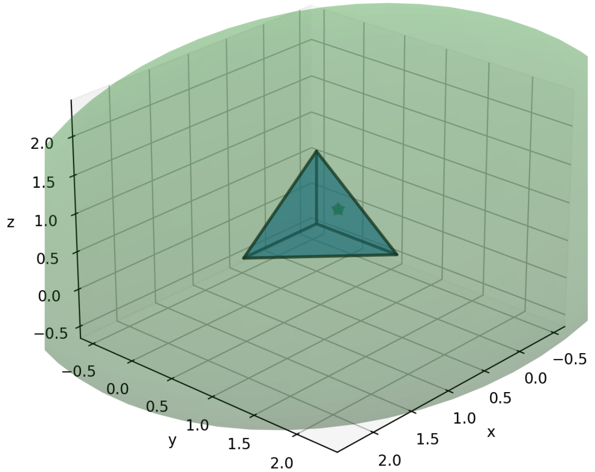
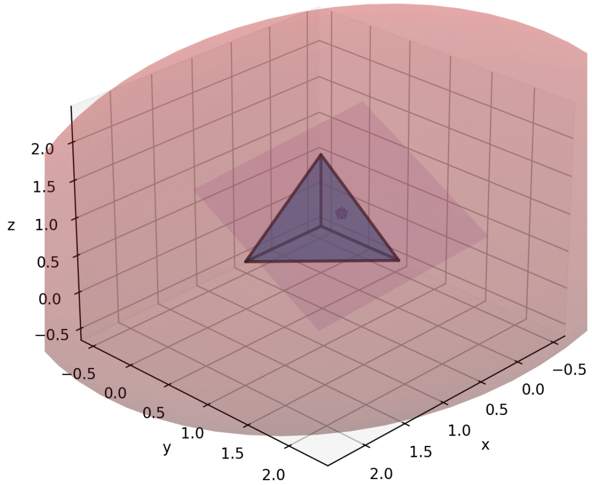


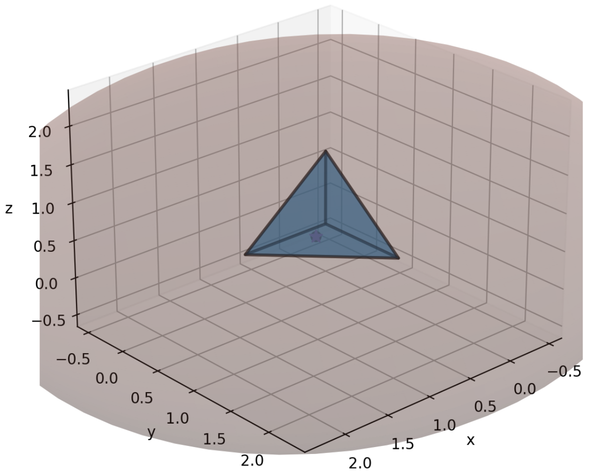
Ellipsoid Method (3D) — iter 7 (show ellipsoid)
feasible center | objective cut | max-axis≈3.183 | best d^T x≈0.800



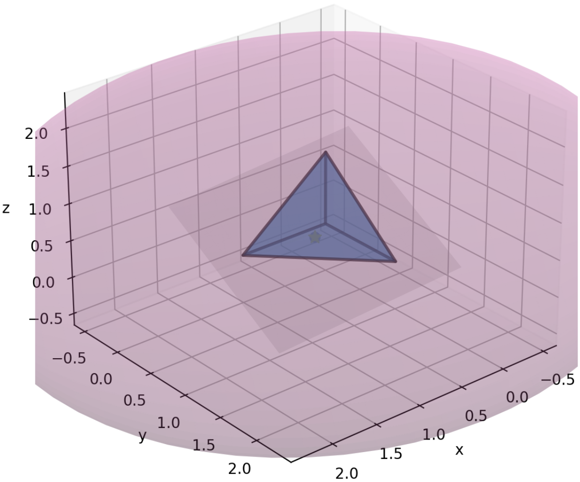
Ellipsoid Method (3D) — iter 7 (show cut plane)
feasible center | objective cut | max-axis≈3.183 | best d^T x≈0.800



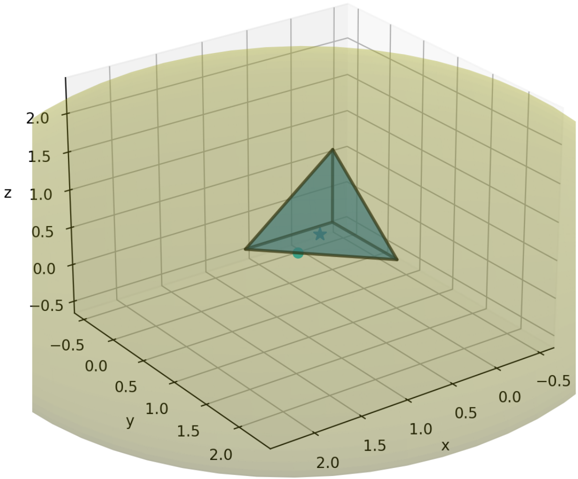
Ellipsoid Method (3D) — iter 8 (show ellipsoid)
feasible center | objective cut | max-axis≈3.240 | best d^T x≈0.344



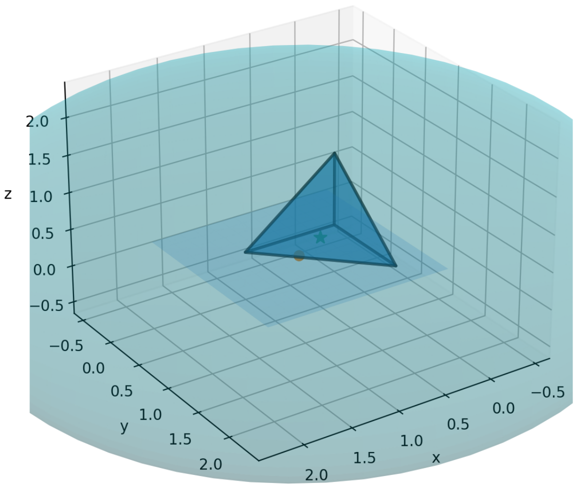
Ellipsoid Method (3D) — iter 8 (show cut plane)
feasible center | objective cut | max-axis≈3.240 | best d^T x≈0.344



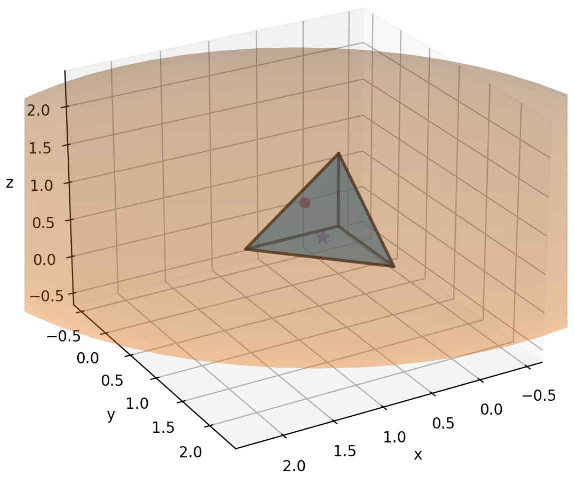
Ellipsoid Method (3D) — iter 9 (show ellipsoid)
infeasible center | constraint cut | max-axis≈3.375 | best d^T x≈0.344



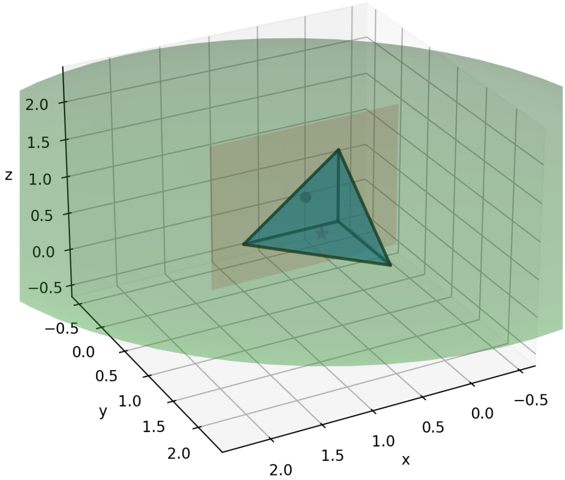
Ellipsoid Method (3D) — iter 9 (show cut plane)
infeasible center | constraint cut | max-axis≈3.375 | best d^T x≈0.344



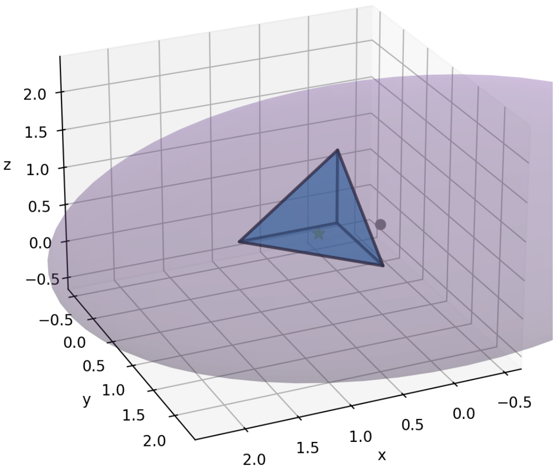
Ellipsoid Method (3D) — iter 10 (show ellipsoid)
infeasible center | constraint cut | max-axis≈3.579 | best d^T x≈0.344



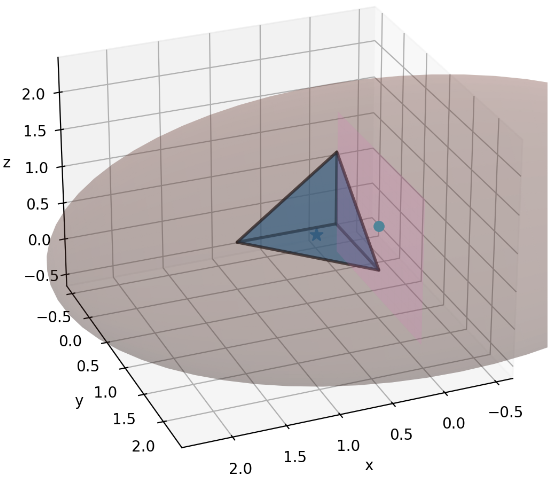
Ellipsoid Method (3D) — iter 10 (show cut plane)
infeasible center | constraint cut | max-axis≈3.579 | best d^T x≈0.344



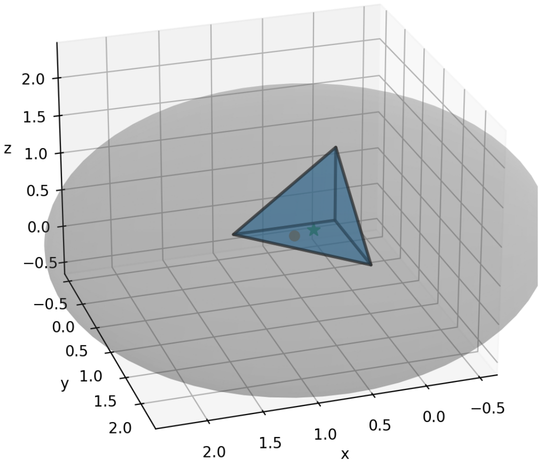
Ellipsoid Method (3D) — iter 11 (show ellipsoid)
infeasible center | constraint cut | max-axis≈3.162 | best d^T x≈0.344



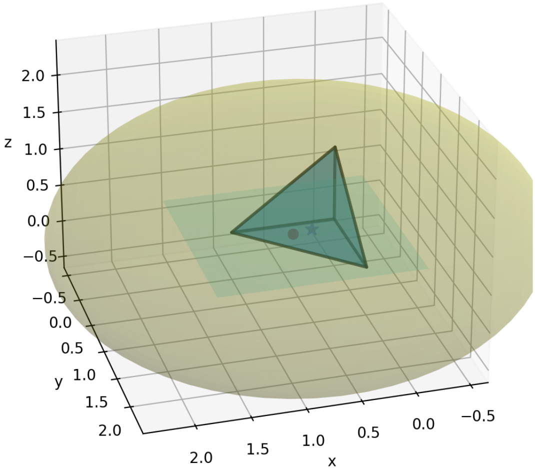
Ellipsoid Method (3D) — iter 11 (show cut plane)
infeasible center | constraint cut | max-axis≈3.162 | best d^T x≈0.344



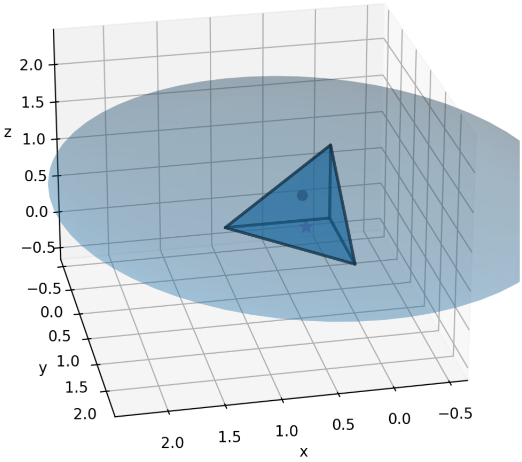
Ellipsoid Method (3D) — iter 12 (show ellipsoid)
infeasible center | constraint cut | max-axis≈2.399 | best d^T x≈0.344



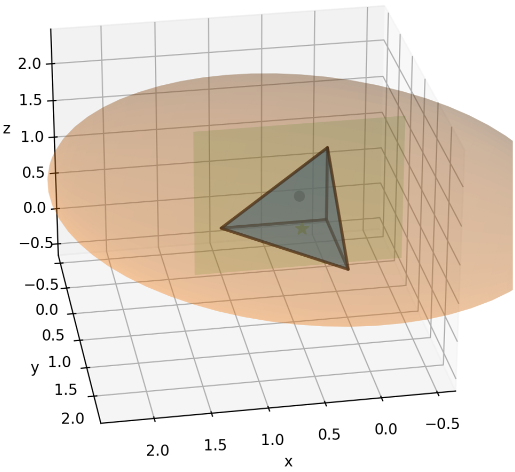
Ellipsoid Method (3D) — iter 12 (show cut plane)
infeasible center | constraint cut | max-axis≈2.399 | best d^T x≈0.344



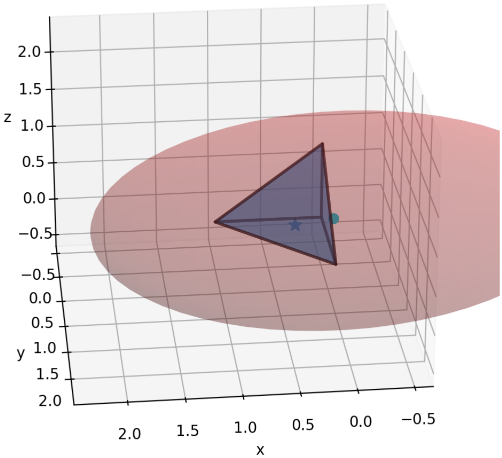
Ellipsoid Method (3D) — iter 13 (show ellipsoid)
infeasible center | constraint cut | max-axis≈2.535 | best d^T x≈0.344



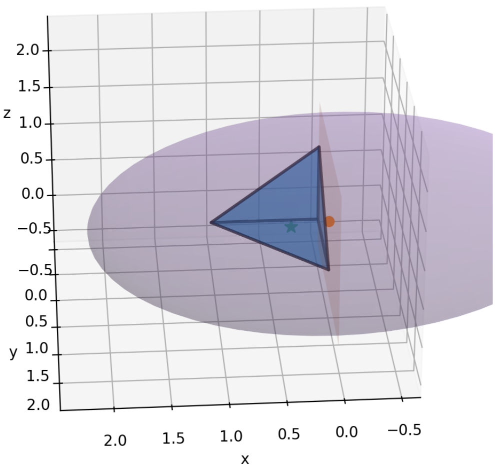
Ellipsoid Method (3D) — iter 13 (show cut plane)
infeasible center | constraint cut | max-axis≈2.535 | best d^T x≈0.344



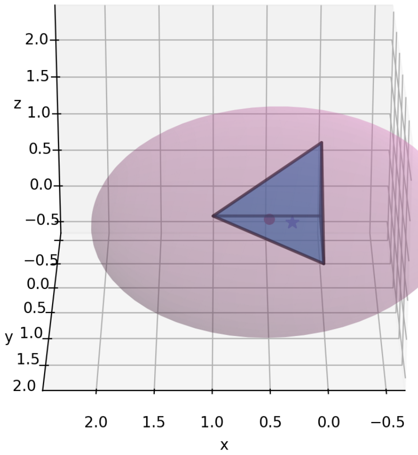
Ellipsoid Method (3D) — iter 14 (show ellipsoid)
infeasible center | constraint cut | max-axis≈2.341 | best d^T x≈0.344



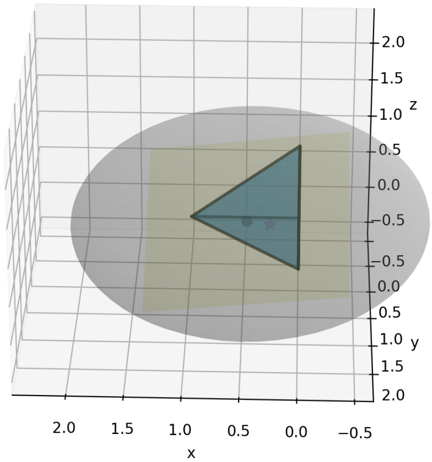
Ellipsoid Method (3D) — iter 14 (show cut plane)
infeasible center | constraint cut | max-axis≈2.341 | best d^T x≈0.344



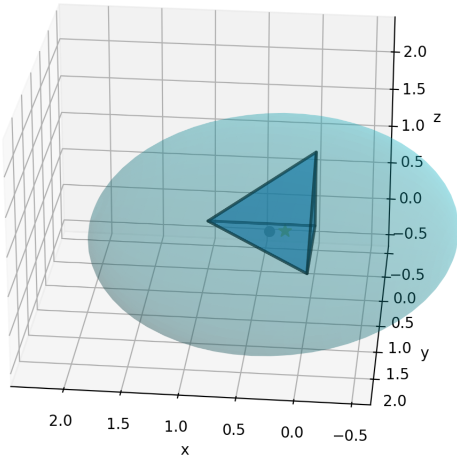
Ellipsoid Method (3D) — iter 15 (show ellipsoid)
feasible center | objective cut | max-axis≈1.772 | best d^T x≈0.344



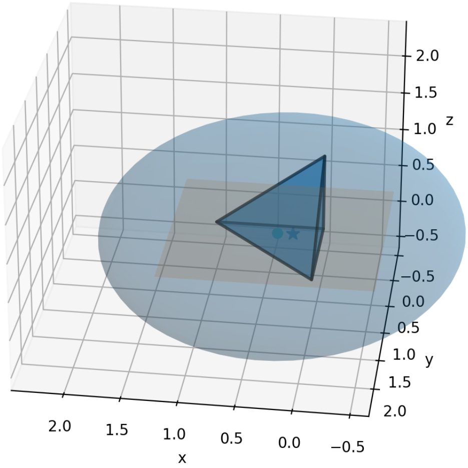
Ellipsoid Method (3D) — iter 15 (show cut plane)
feasible center | objective cut | max-axis≈1.772 | best d^T x≈0.344



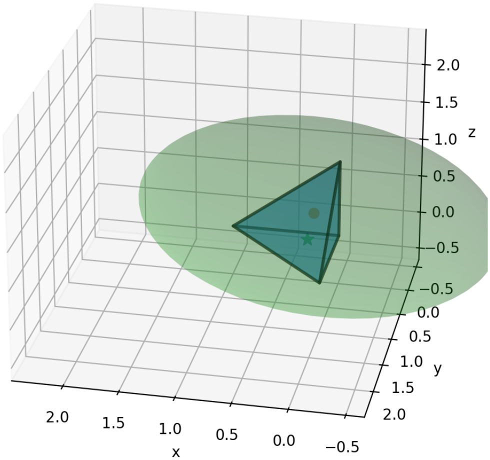
Ellipsoid Method (3D) — iter 16 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.879 | best d^T x≈0.344



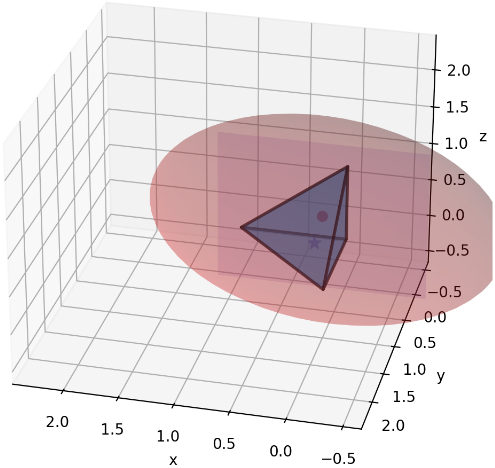
Ellipsoid Method (3D) — iter 16 (show cut plane)
infeasible center | constraint cut | max-axis≈1.879 | best d^T x≈0.344



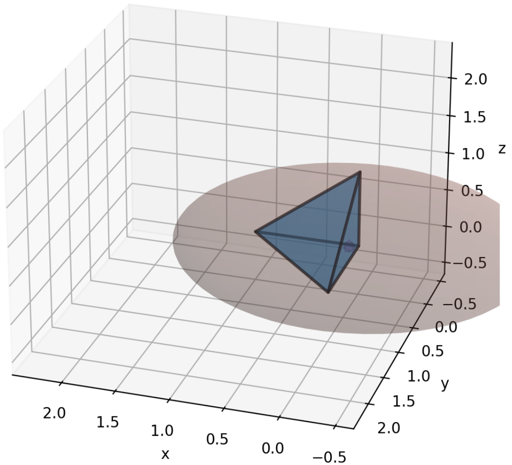
Ellipsoid Method (3D) — iter 17 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.978 | best d^T x≈0.344



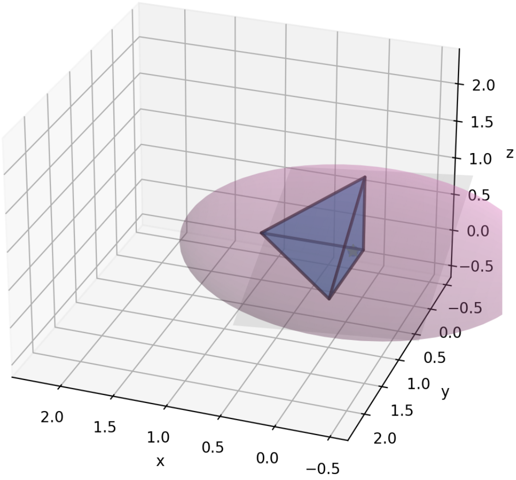
Ellipsoid Method (3D) — iter 17 (show cut plane)
infeasible center | constraint cut | max-axis≈1.978 | best d^T x≈0.344



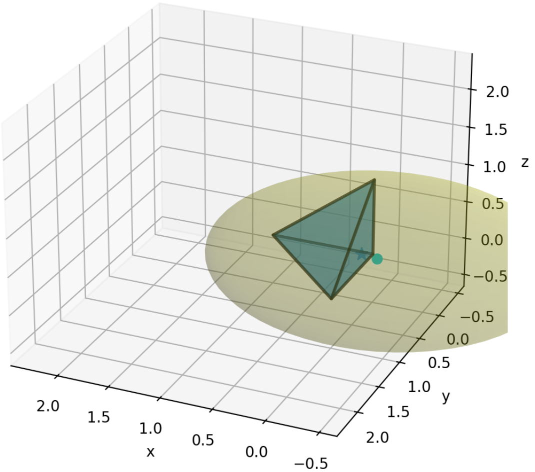
Ellipsoid Method (3D) — iter 18 (show ellipsoid)
feasible center | objective cut | max-axis≈1.876 | best d^T x≈0.318



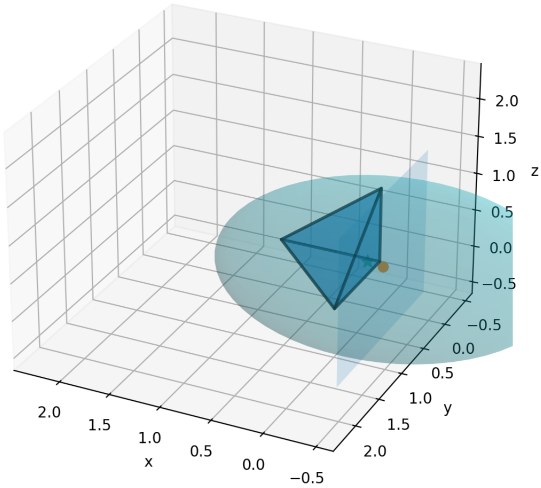
Ellipsoid Method (3D) — iter 18 (show cut plane)
feasible center | objective cut | max-axis≈1.876 | best d^T x≈0.318



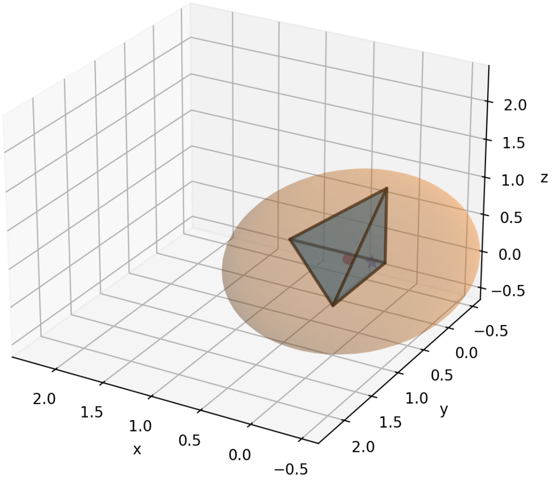
Ellipsoid Method (3D) — iter 19 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.971 | best d^T x≈0.318

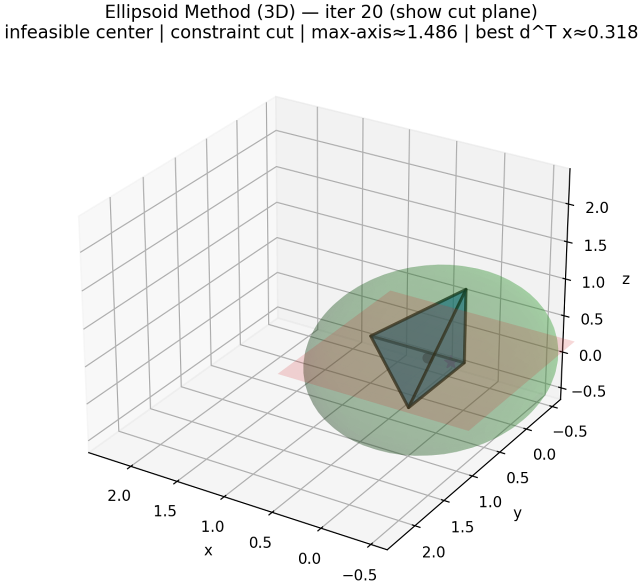


Ellipsoid Method (3D) — iter 19 (show cut plane)
infeasible center | constraint cut | max-axis≈1.971 | best d^T x≈0.318

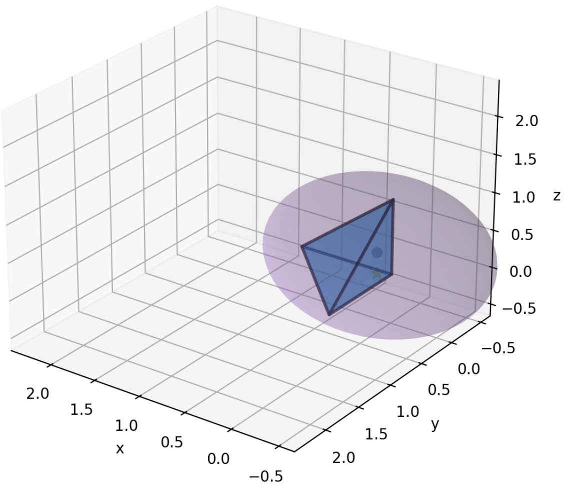


Ellipsoid Method (3D) — iter 20 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.486 | best d^T x≈0.318

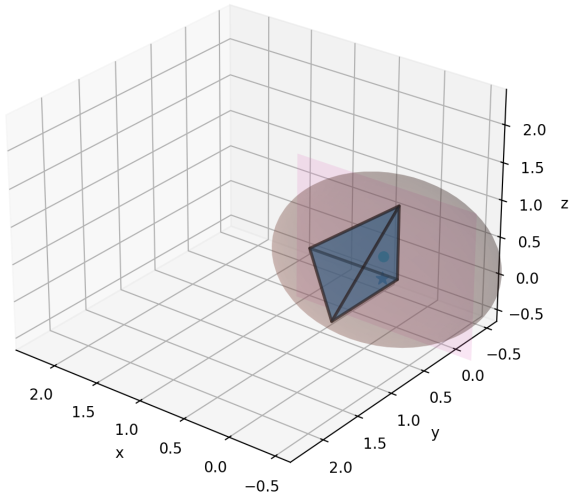



Ellipsoid Method (3D) — iter 20 (show cut plane)
infeasible center | constraint cut | max-axis≈1.486 | best d^T x≈0.318
2.0
1.5
1.0
0.5
0.0
−0.5
z
−0.5
0.0
0.5
1.0
1.5
2.0
y
2.0
1.5
1.0
0.5
0.0
−0.5
x

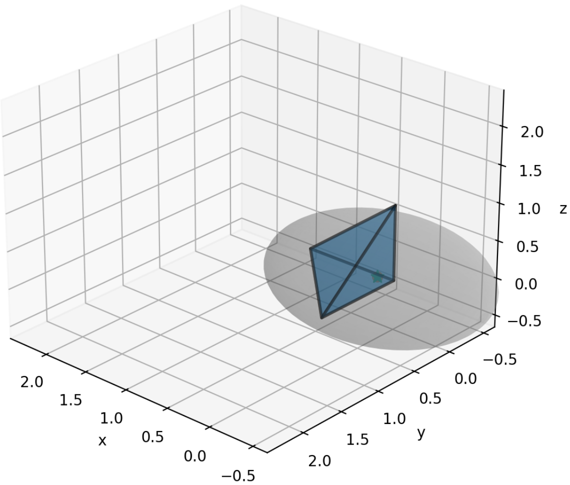
Ellipsoid Method (3D) — iter 21 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.557 | best d^T x≈0.318



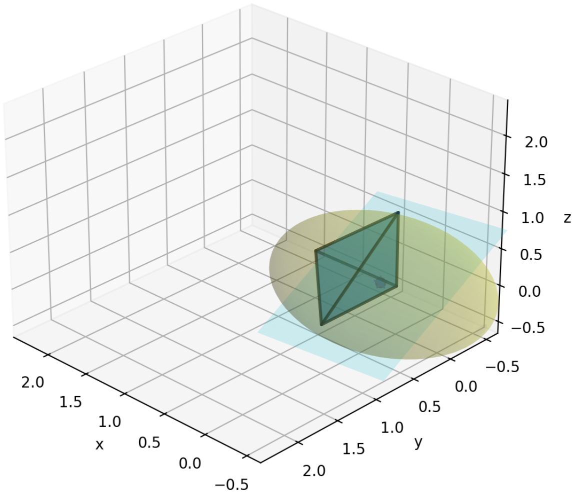
Ellipsoid Method (3D) — iter 21 (show cut plane)
infeasible center | constraint cut | max-axis≈1.557 | best d^T x≈0.318



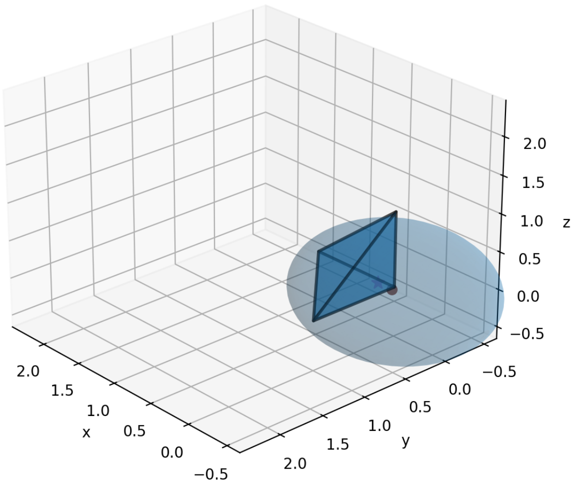
Ellipsoid Method (3D) — iter 22 (show ellipsoid)
feasible center | objective cut | max-axis≈1.499 | best d^T x≈0.252



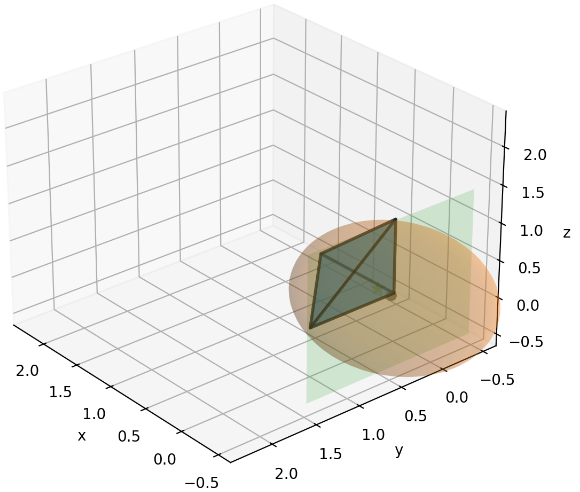
Ellipsoid Method (3D) — iter 22 (show cut plane)
feasible center | objective cut | max-axis≈1.499 | best d^T x≈0.252



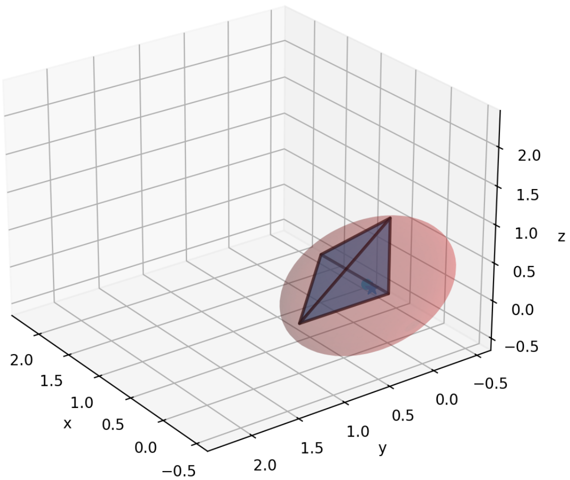
Ellipsoid Method (3D) — iter 23 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.564 | best d^T x≈0.252



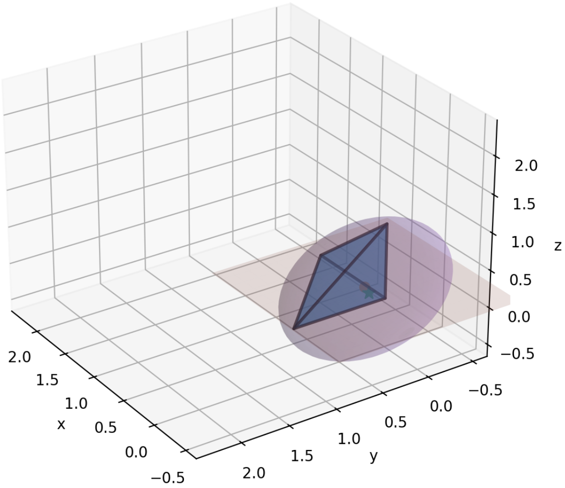
Ellipsoid Method (3D) — iter 23 (show cut plane)
infeasible center | constraint cut | max-axis≈1.564 | best d^T x≈0.252



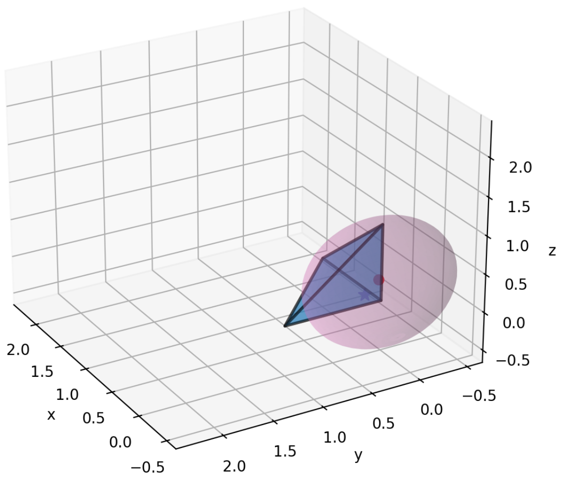
Ellipsoid Method (3D) — iter 24 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.178 | best d^T x≈0.252

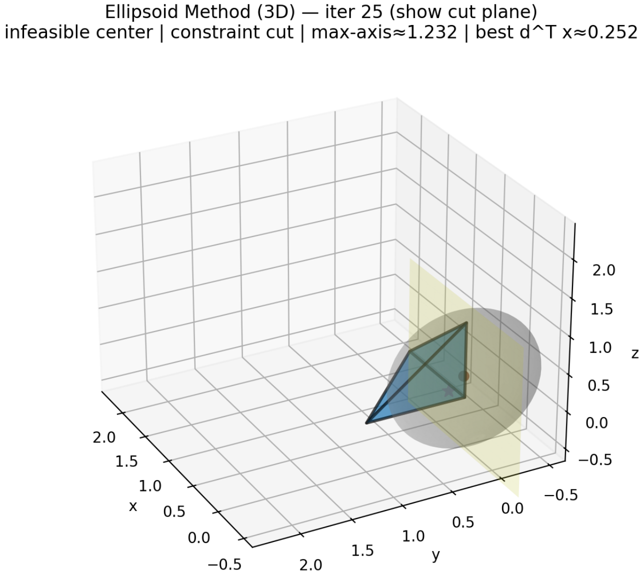


Ellipsoid Method (3D) — iter 24 (show cut plane)
infeasible center | constraint cut | max-axis≈1.178 | best d^T x≈0.252

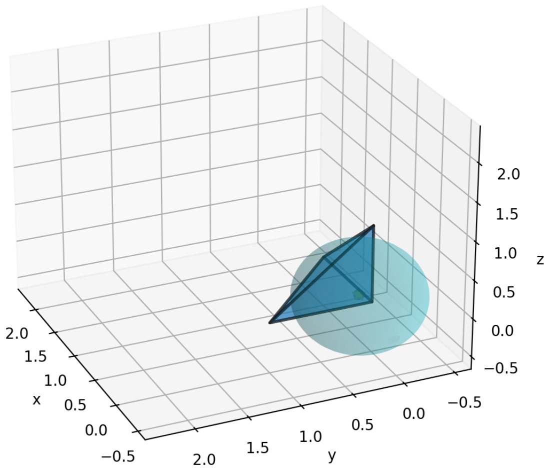


Ellipsoid Method (3D) — iter 25 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.232 | best d^T x≈0.252

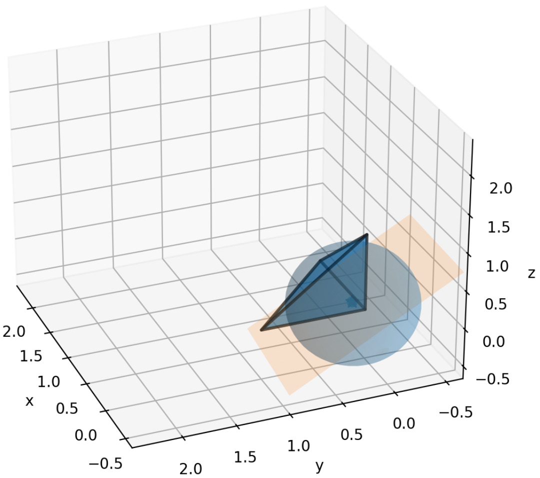



Ellipsoid Method (3D) — iter 25 (show cut plane)
infeasible center | constraint cut | max-axis≈1.232 | best d^T x≈0.252
x
y
z
2.0
1.5
1.0
0.5
0.0
−0.5

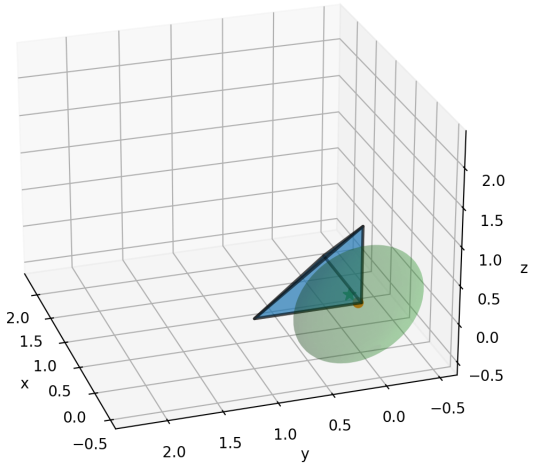
Ellipsoid Method (3D) — iter 26 (show ellipsoid)
feasible center | objective cut | max-axis≈1.196 | best d^T x≈0.205



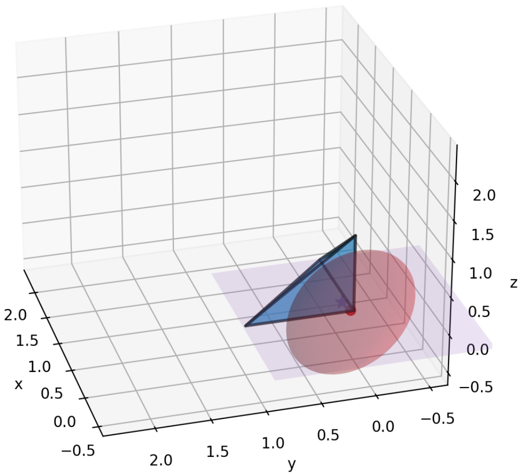
Ellipsoid Method (3D) — iter 26 (show cut plane)
feasible center | objective cut | max-axis≈1.196 | best d^T x≈0.205



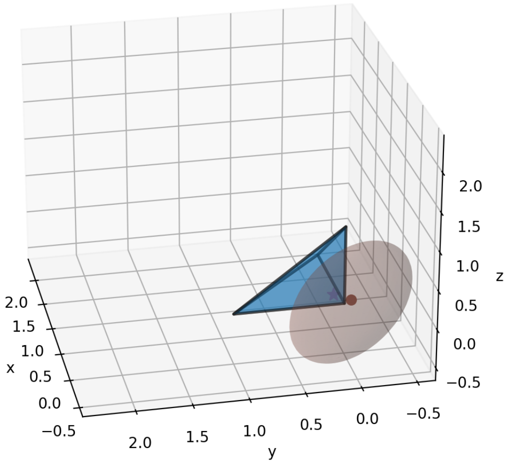
Ellipsoid Method (3D) — iter 27 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.242 | best d^T x≈0.205



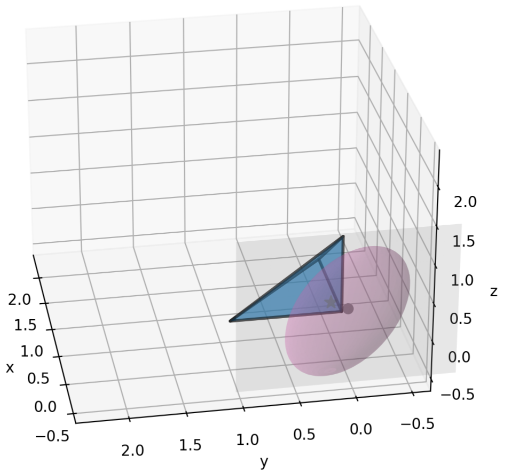
Ellipsoid Method (3D) — iter 27 (show cut plane)
infeasible center | constraint cut | max-axis≈1.242 | best d^T x≈0.205



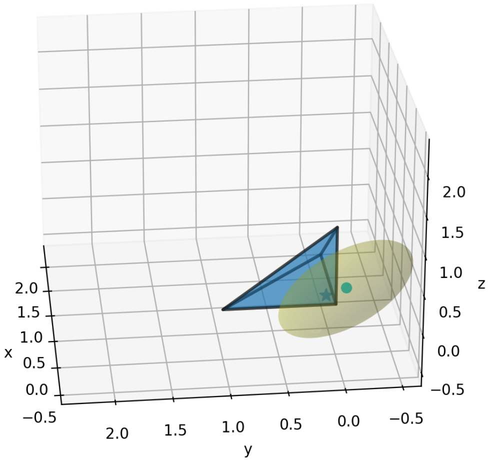
Ellipsoid Method (3D) — iter 28 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.253 | best d^T x≈0.205



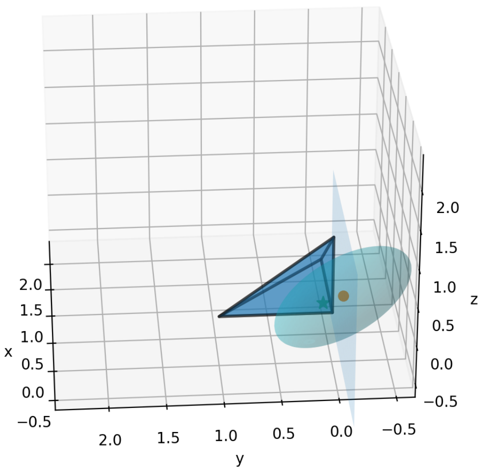
Ellipsoid Method (3D) — iter 28 (show cut plane)
infeasible center | constraint cut | max-axis≈1.253 | best d^T x≈0.205



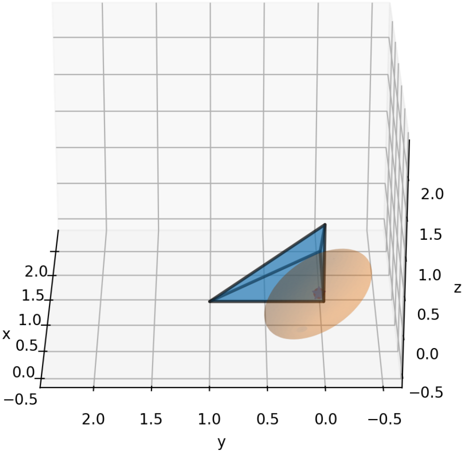
Ellipsoid Method (3D) — iter 29 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.951 | best d^T x≈0.205



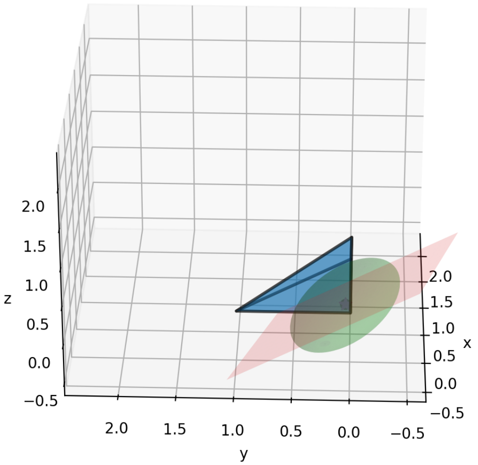
Ellipsoid Method (3D) — iter 29 (show cut plane)
infeasible center | constraint cut | max-axis≈0.951 | best d^T x≈0.205



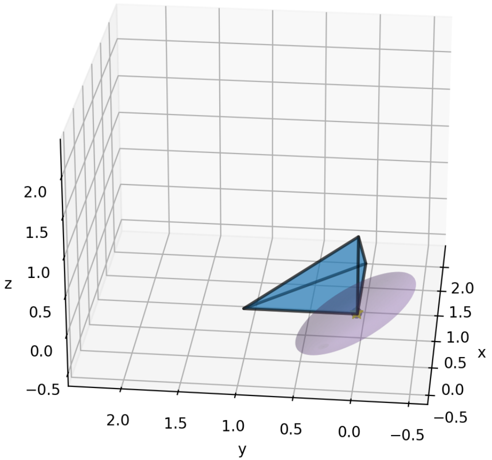
Ellipsoid Method (3D) — iter 30 (show ellipsoid)
feasible center | objective cut | max-axis≈0.932 | best d^T x≈0.170



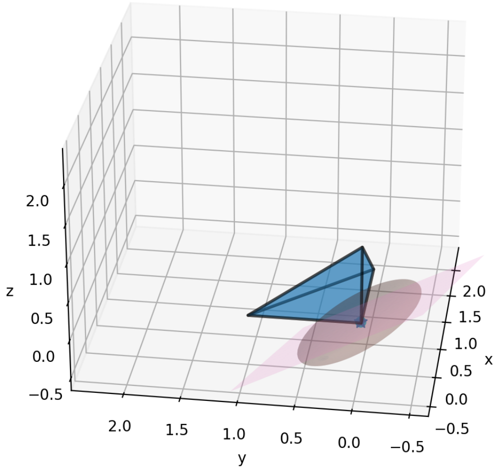
Ellipsoid Method (3D) — iter 30 (show cut plane)
feasible center | objective cut | max-axis≈0.932 | best d^T x≈0.170



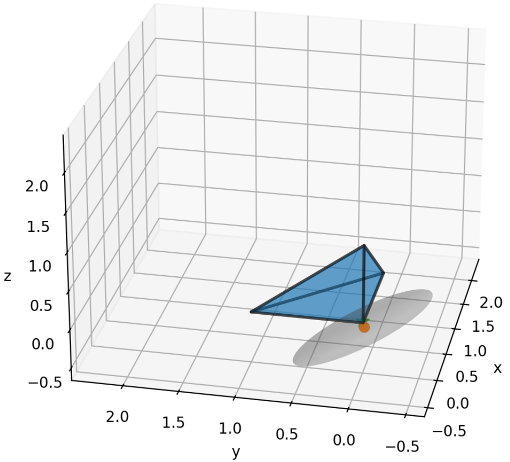
Ellipsoid Method (3D) — iter 31 (show ellipsoid)
feasible center | objective cut | max-axis≈0.968 | best d^T x≈0.034



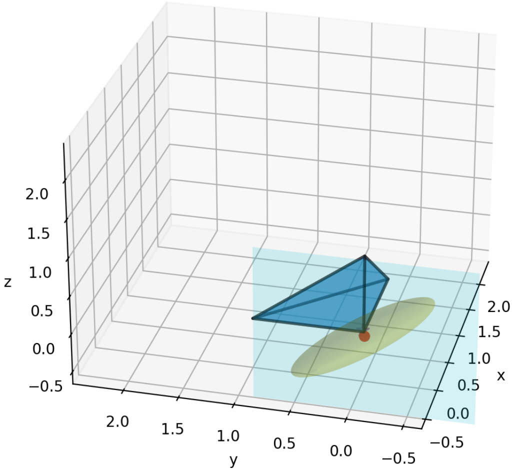
Ellipsoid Method (3D) — iter 31 (show cut plane)
feasible center | objective cut | max-axis≈0.968 | best d^T x≈0.034

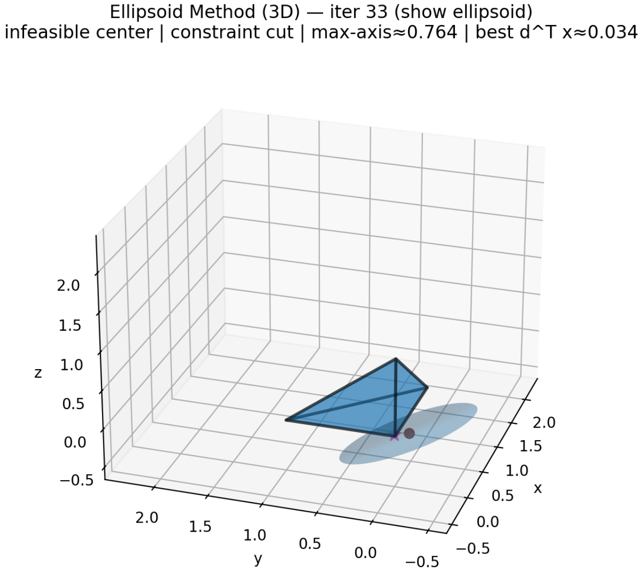


Ellipsoid Method (3D) — iter 32 (show ellipsoid)
infeasible center | constraint cut | max-axis≈1.016 | best d^T x≈0.034

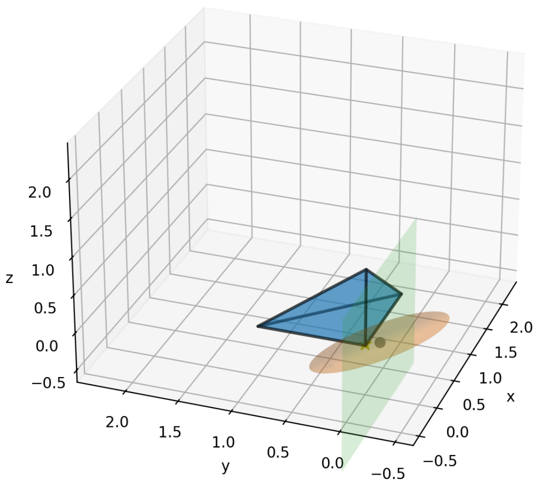


Ellipsoid Method (3D) — iter 32 (show cut plane)
infeasible center | constraint cut | max-axis≈1.016 | best d^T x≈0.034

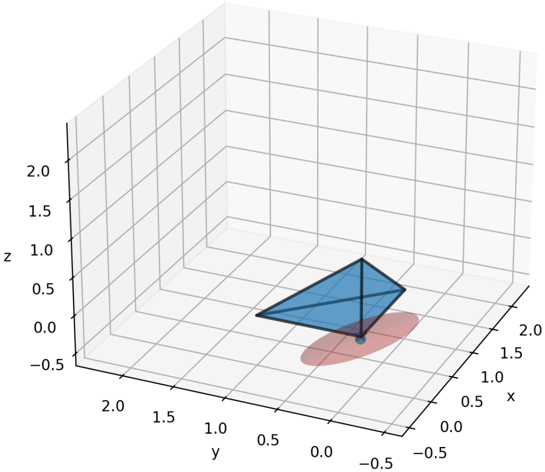



Ellipsoid Method (3D) — iter 33 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.764 | best d^T x≈0.034
z
2.0
1.5
1.0
0.5
0.0
−0.5
y
2.0
1.5
1.0
0.5
0.0
−0.5
x
2.0
1.5
1.0
0.5
0.0
−0.5

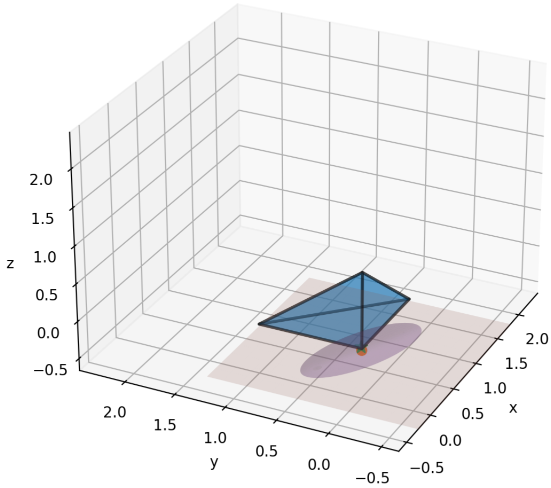
Ellipsoid Method (3D) — iter 33 (show cut plane)
infeasible center | constraint cut | max-axis≈0.764 | best d^T x≈0.034



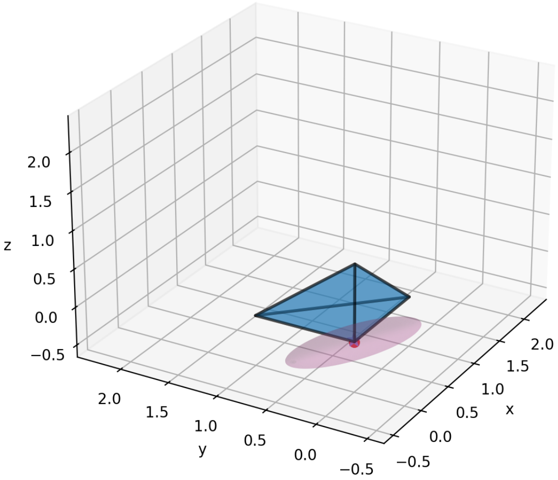
Ellipsoid Method (3D) — iter 34 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.790 | best d^T x≈0.034



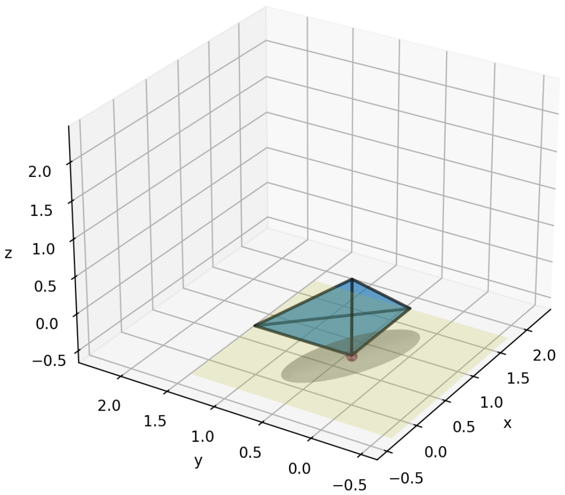
Ellipsoid Method (3D) — iter 34 (show cut plane)
infeasible center | constraint cut | max-axis≈0.790 | best d^T x≈0.034



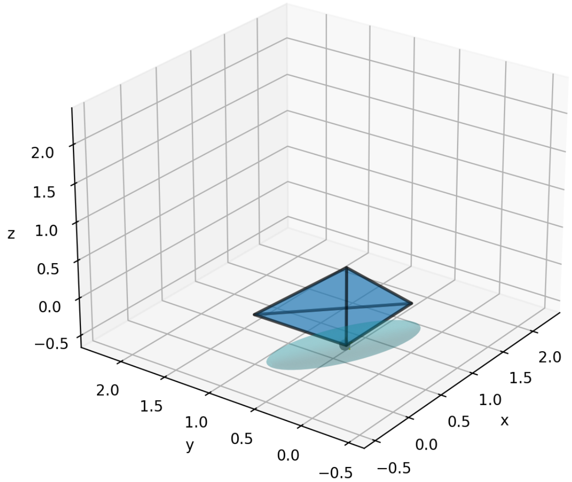
Ellipsoid Method (3D) — iter 35 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.764 | best d^T x≈0.034



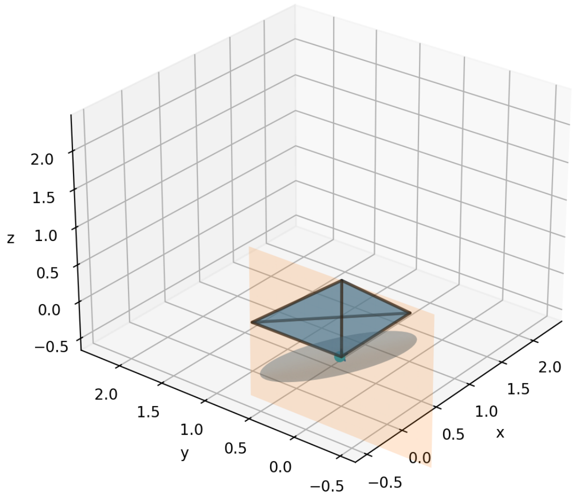
Ellipsoid Method (3D) — iter 35 (show cut plane)
infeasible center | constraint cut | max-axis≈0.764 | best d^T x≈0.034



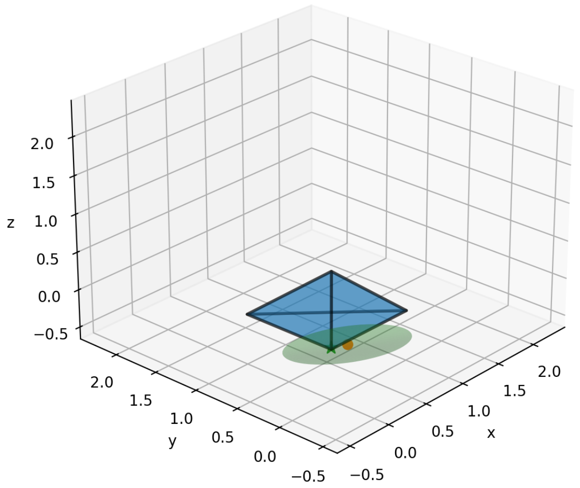
Ellipsoid Method (3D) — iter 36 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.780 | best d^T x≈0.034



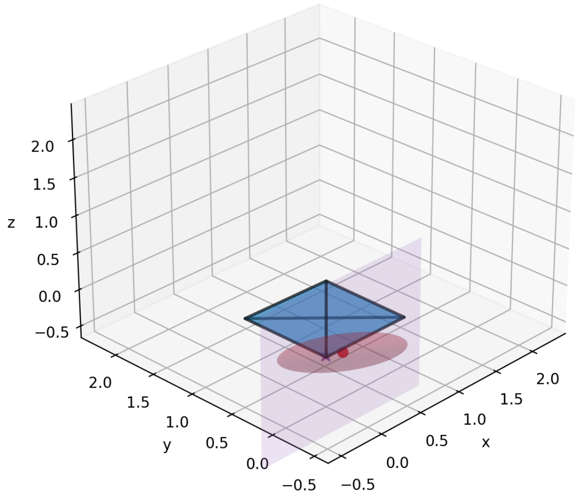
Ellipsoid Method (3D) — iter 36 (show cut plane)
infeasible center | constraint cut | max-axis≈0.780 | best d^T x≈0.034



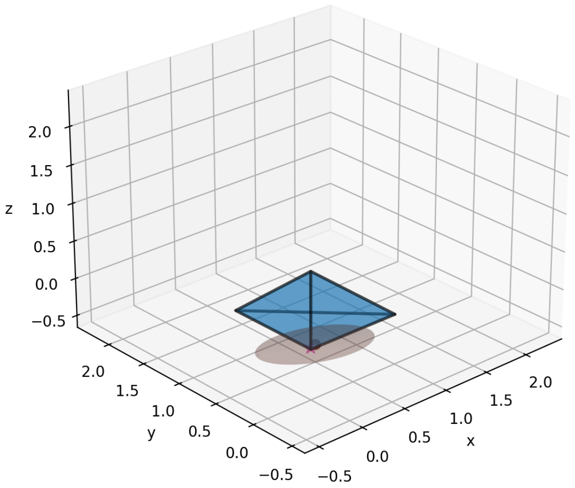
Ellipsoid Method (3D) — iter 37 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.593 | best d^T x≈0.034



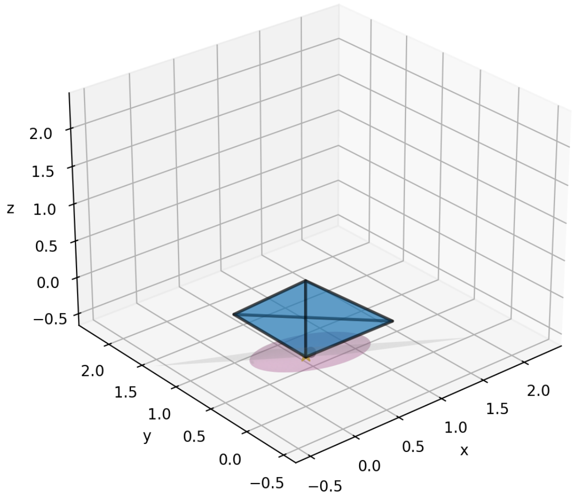
Ellipsoid Method (3D) — iter 37 (show cut plane)
infeasible center | constraint cut | max-axis≈0.593 | best d^T x≈0.034



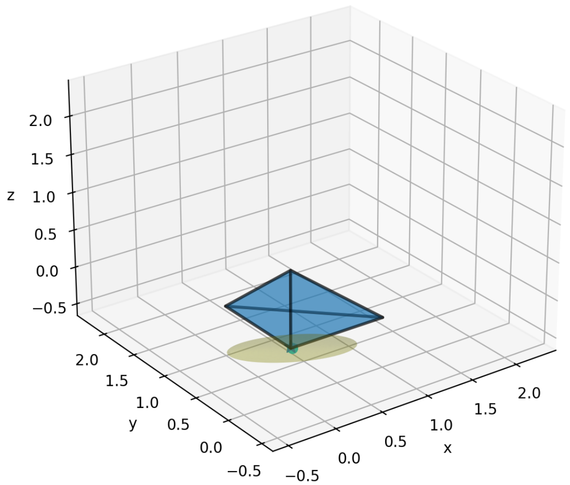
Ellipsoid Method (3D) — iter 38 (show ellipsoid)
feasible center | objective cut | max-axis≈0.582 | best d^T x≈0.034



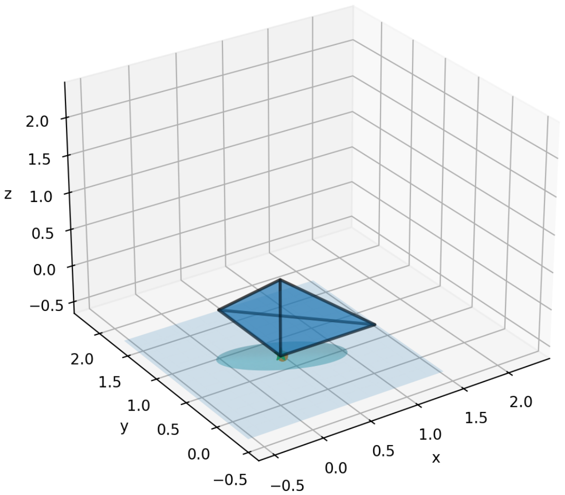
Ellipsoid Method (3D) — iter 38 (show cut plane)
feasible center | objective cut | max-axis≈0.582 | best d^T x≈0.034



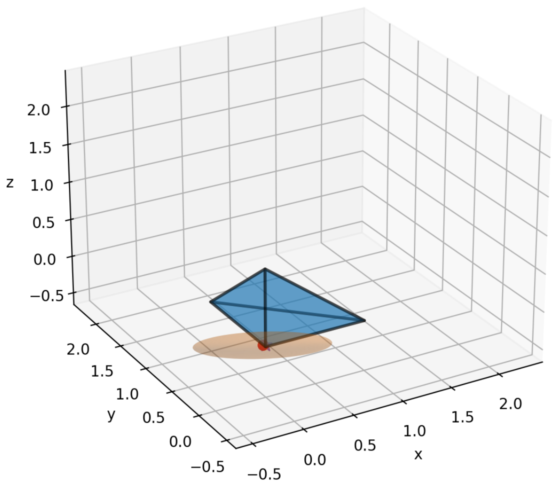
Ellipsoid Method (3D) — iter 39 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.600 | best d^T x≈0.034

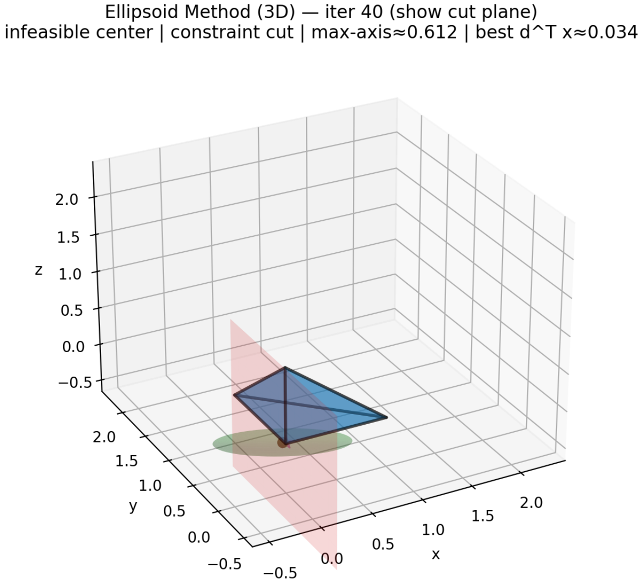


Ellipsoid Method (3D) — iter 39 (show cut plane)
infeasible center | constraint cut | max-axis≈0.600 | best d^T x≈0.034

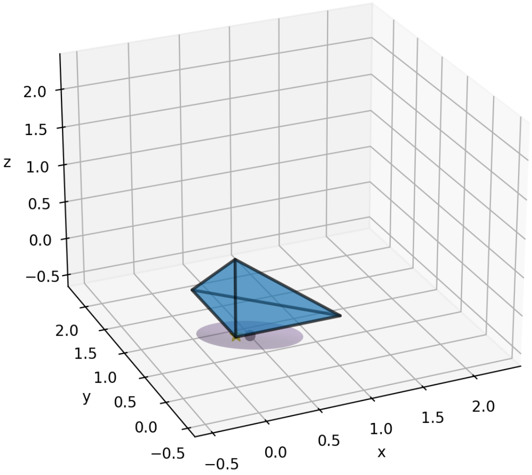


Ellipsoid Method (3D) — iter 40 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.612 | best d^T x≈0.034

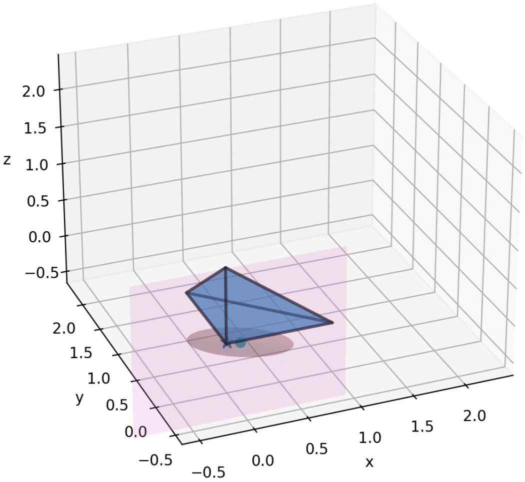



Ellipsoid Method (3D) — iter 40 (show cut plane)
infeasible center | constraint cut | max-axis≈0.612 | best d^T x≈0.034
z
2.0
1.5
1.0
0.5
0.0
−0.5
y
2.0
1.5
1.0
0.5
0.0
−0.5
x
−0.5
0.0
0.5
1.0
1.5
2.0

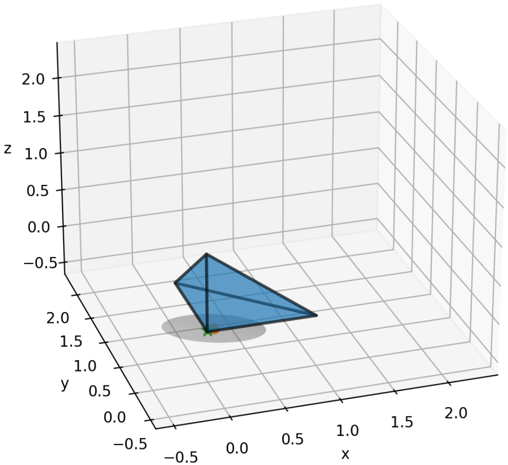
Ellipsoid Method (3D) — iter 41 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.465 | best d^T x≈0.034



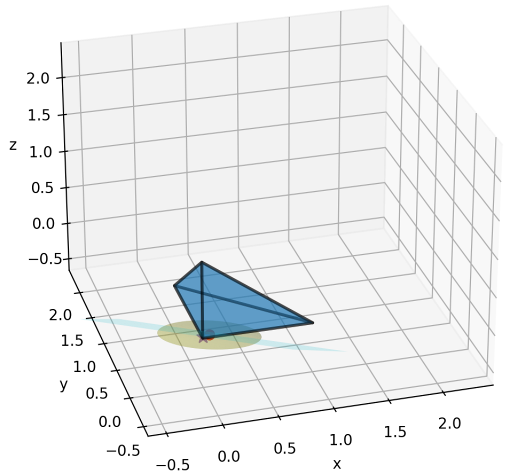
Ellipsoid Method (3D) — iter 41 (show cut plane)
infeasible center | constraint cut | max-axis≈0.465 | best d^T x≈0.034



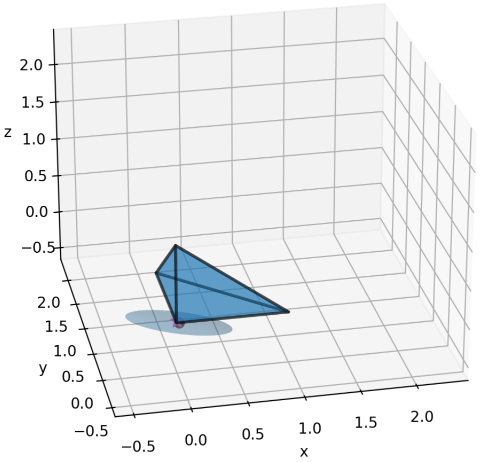
Ellipsoid Method (3D) — iter 42 (show ellipsoid)
feasible center | objective cut | max-axis≈0.459 | best d^T x≈0.034



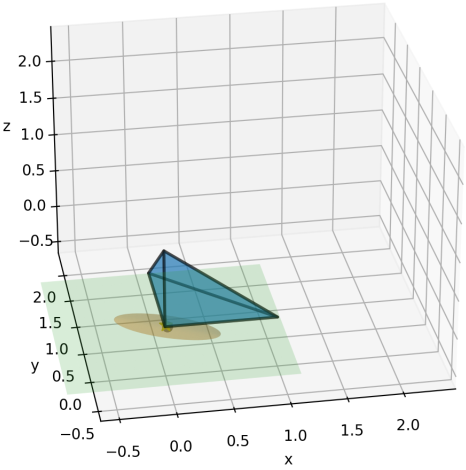
Ellipsoid Method (3D) — iter 42 (show cut plane)
feasible center | objective cut | max-axis≈0.459 | best d^T x≈0.034



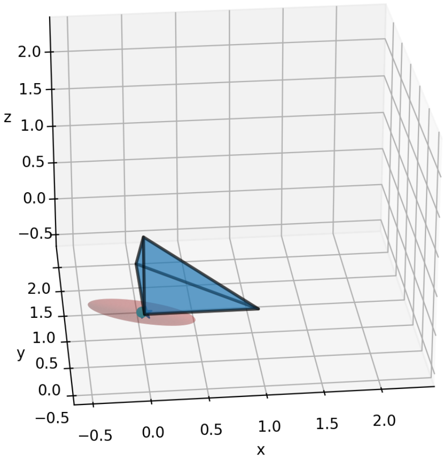
Ellipsoid Method (3D) — iter 43 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.474 | best d^T x≈0.034



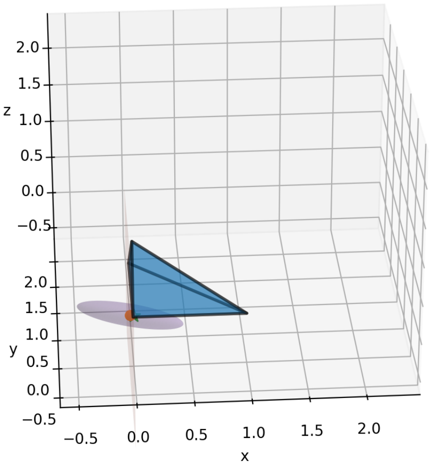
Ellipsoid Method (3D) — iter 43 (show cut plane)
infeasible center | constraint cut | max-axis≈0.474 | best d^T x≈0.034



Ellipsoid Method (3D) — iter 44 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.482 | best d^T x≈0.034



Ellipsoid Method (3D) — iter 44 (show cut plane)
infeasible center | constraint cut | max-axis≈0.482 | best d^T x≈0.034



Ellipsoid Method (3D) — iter 45 (show ellipsoid)
infeasible center | constraint cut | max-axis≈0.366 | best d^T x≈0.034