CS498: Algorithmic Engineering

Lecture 9: Large Scale Integer Linear Programs, Lazy Cuts, User Cuts.

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Week 05 – 02/17/2026

Outline

1. Recap: Branch and Bound & LP Relaxation Strength

2. Case Study (IP): Directed TSP, Lazy Constraints and User Cuts

3. Practical Tips and Summary

Branch and Bound: The Big Picture

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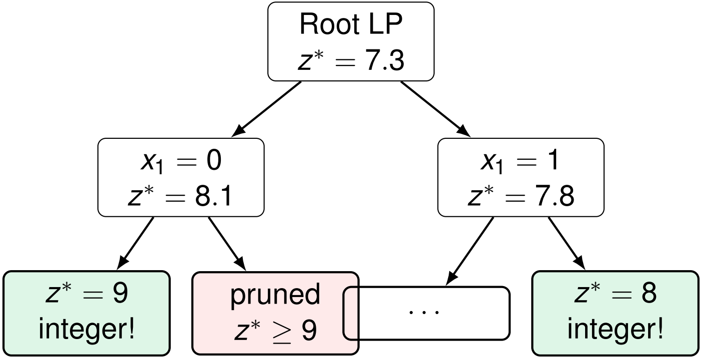

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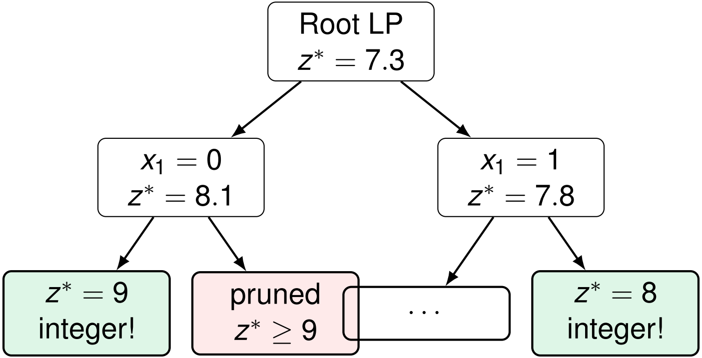

At **every node**: solve an LP relaxation to get a bound, then branch or prune.

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Takeaway: The quality of the LP relaxation largely determines how fast B&B converges.

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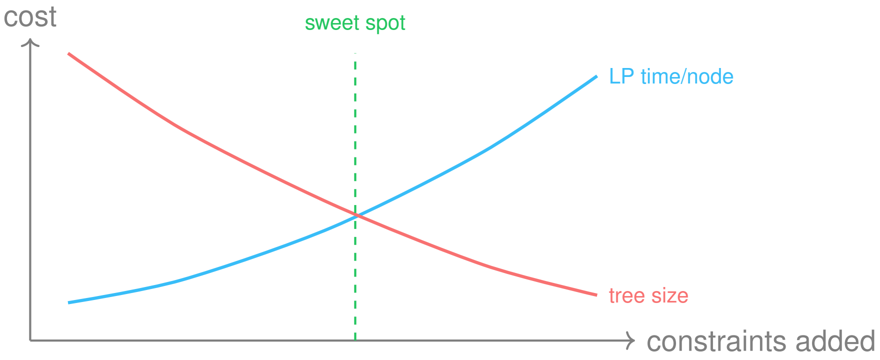

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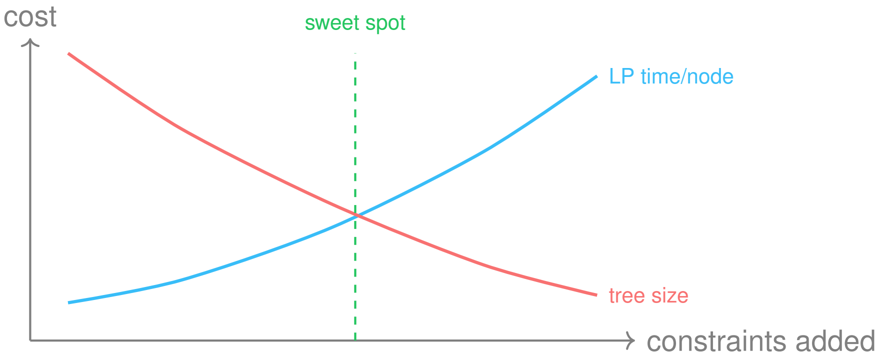

Tradeoff: More constraints $\Rightarrow$ tighter LP $\Rightarrow$ fewer nodes. But each LP solve takes longer.

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(B) Fractional node LP solved

Solution $x \in [0, 1]^n$.
Check: any violated inequality?
If yes $\Rightarrow$ tighten via `cbCut`.

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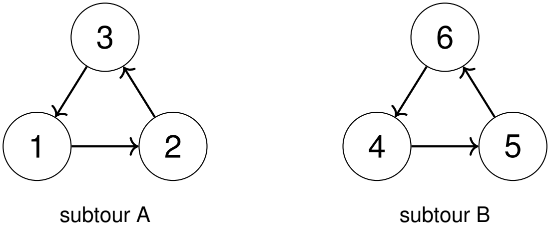
But

They do *not* enforce one single tour. They allow multiple disjoint cycles (subtours).

Picture: Degree Constraints Can Produce Multiple Cycles



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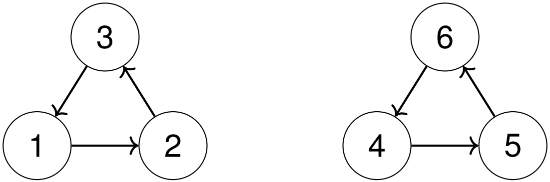

subtour A

subtour B

We need to **forbid subtours**.

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So why not just use MTZ again?

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We need a formulation whose LP relaxation is **much tighter**.

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The deal: we trade a compact-but-weak LP formulation for a tight-but-huge one, and use **row generation** to make it tractable.

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So we will **generate them on demand**. But how?

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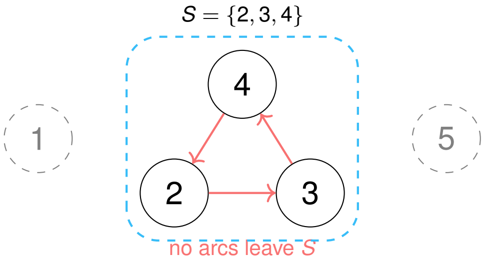

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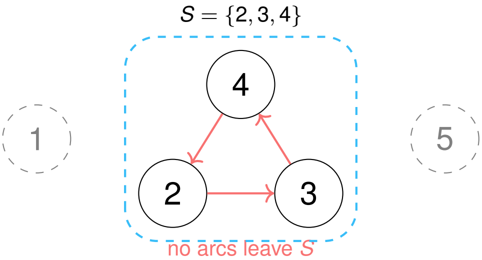

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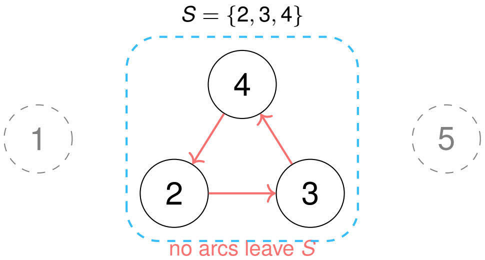

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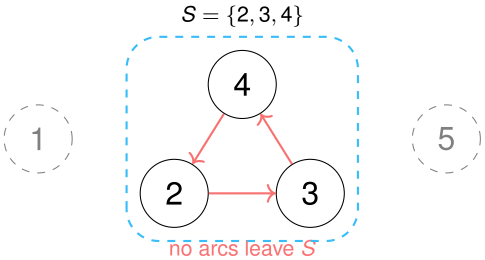

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A single Hamiltonian tour always has at least one arc leaving every proper subset $\Rightarrow$ satisfied.

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The Lecture 8 approach: row generation, start small, add violated constraints iteratively. But now we are inside an IP, not just an LP...

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Gurobi's mechanism: callbacks, or functions we write that Gurobi calls at specific moments during the solve.

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This is **exactly** what `cbLazy` does. You are literally being "lazy" about the constraints, only providing them when Gurobi asks.

And Gurobi is surprisingly okay with this arrangement. It just needs you to set one flag: `LazyConstraints = 1`.

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This is exactly the `cbLazy` mechanism.

Separation Oracle for Integer Solutions (Easy!)

Given an integer solution $x^* \in \{0, 1\}^{|E|}$, how do we detect if there is a subtour or not?

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Complexity: $O(n+m)$: linear time. Very fast.

Gurobi Callbacks: How They Work

A **callback** is a Python function you write that Gurobi calls at specific moments during the solve.

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```
def callback(model, where):
    # 'where' tells you WHAT just happened inside the solver
    if where == GRB.Callback.MIPSOL:
        # An integer solution was just found!
        ...
    elif where == GRB.Callback.MIPNODE:
        # A node LP relaxation was just solved!
        ...

m.optimize(callback)   # pass your function to optimize()
```

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Key idea: `where` is an event type. You react only to the events you care about.

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```
# The integer solution values
xsol = model.cbGetSolution(model._x)

# Objective value of this incumbent
obj = model.cbGet(GRB.Callback.MIPSOL_OBJ)

# Which B&B node found it
node = model.cbGet(GRB.Callback.MIPSOL_NODCNT)

# Number of solutions found so far
solcnt = model.cbGet(GRB.Callback.MIPSOL_SOLCNT)
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What you can do:

```
# Reject this solution by adding a constraint
model.cbLazy( <linear expression> >= <rhs> )
```

Subtour Detection Code

```python
import networkx as nx

def find_subtours(n, xsol):
    """Given integer solution (all 0.0 or 1.0), return list of subtour node sets."""
    G = nx.DiGraph()
    G.add_nodes_from(range(n))

    for i in range(n):
        for j in range(n):
            if i != j and int(round(xsol[i, j]))==1:  # round 1.0 to 1 and 0.0 to 0
                G.add_edge(i, j)

    components = list(nx.strongly_connected_components(G))

    # A subtour is any component smaller than n
    return [S for S in components if len(S) < n]
```

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    # A subtour is any component smaller than n
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```

If `find_subtours` returns an empty list ⇒ the solution is a valid tour.

The cbLazy Callback

```python
def callback(model, where):
    if where == GRB.Callback.MIPSOL:
        x = model._x
        n = model._n

        # 1. Read the integer solution
        xsol = model.cbGetSolution(x)

        # 2. Find subtours
        subtours = find_subtours(n, xsol)

        # 3. For each subtour, add a lazy constraint
        for S in subtours:
            complement = set(range(n)) - S
            model.cbLazy(
                gp.quicksum(x[i,j] for i in S for j in complement) >= 1
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```

Each `cbLazy` call adds one DFJ inequality. Gurobi rejects the incumbent and continues searching.

Building the Model (cbLazy Only)

```
import gurobipy as gp
from gurobipy import GRB

m = gp.Model("TSP_lazy_only")
x = m.addVars(V, V, vtype=GRB.BINARY, name="x")
for i in V:
    x[i,i].ub = 0   # no self-loops

m.setObjective(
    gp.quicksum(c[i,j]*x[i,j] for i in V for j in V if i != j),
    GRB.MINIMIZE)

for i in V:
    m.addConstr(gp.quicksum(x[i,j] for j in V if j != i) == 1)
    m.addConstr(gp.quicksum(x[j,i] for j in V if j != i) == 1)

m.Params.LazyConstraints = 1   # REQUIRED: tells Gurobi to expect cbLazy

m._x = x    # attach variables so callback can access them
m._n = n
m.optimize(callback)
```

Important: `LazyConstraints = 1`

If you forget this parameter

```
# m.Params.LazyConstraints = 1   # OOPS, commented out!
m.optimize(callback)
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Gurobi will **silently ignore** all `cbLazy` calls. Your model will return solutions with subtours.

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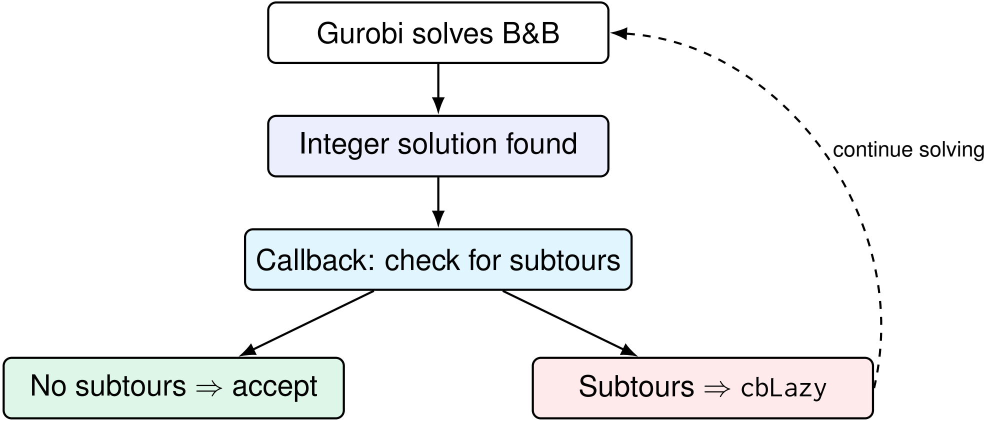
Why does this parameter exist?
When lazy constraints are enabled, Gurobi cannot use certain presolve reductions and heuristics that assume the model is complete. So it needs to know *in advance* that you plan to add constraints during the solve.

cbLazy: What Happens at Runtime



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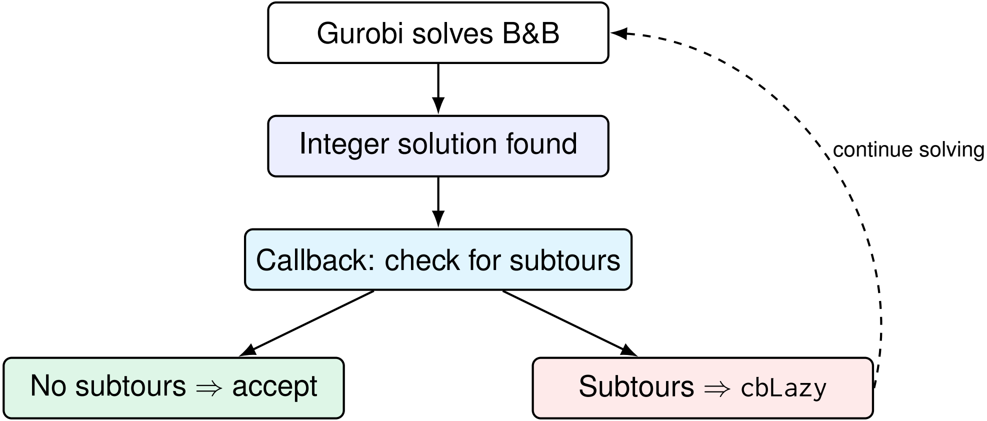

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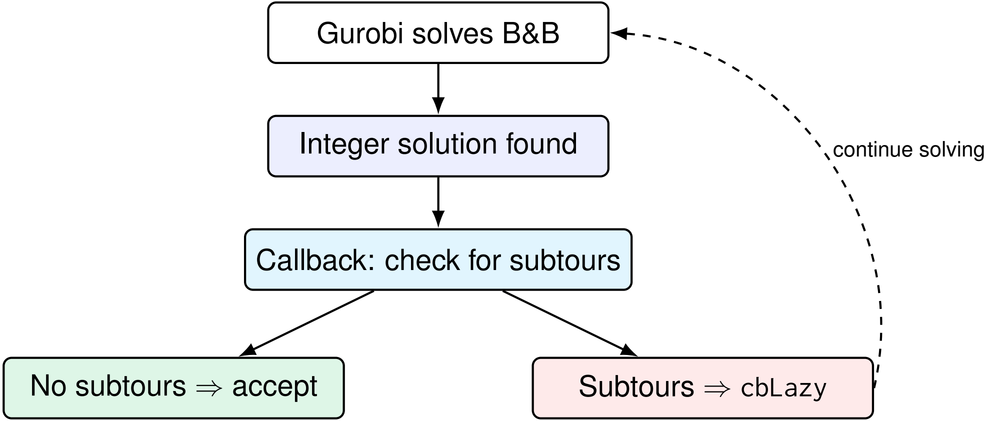

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But is the solver **efficient**?

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Consequence:

- LP bounds are weak $\Rightarrow$ almost no pruning.
- Gurobi explores a **massive** B&B tree.
- The solver finds many integer incumbents with subtours, rejects them via `cbLazy`, but makes slow progress.

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This is the `cbCut` mechanism: adding **user cuts** at fractional node LP solutions.

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- `cbLazy` Building inspector: “This violates code. Tear it down.” (correctness).

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Analogy:

- `cbLazy` Building inspector: “This violates code. Tear it down.” (correctness).
- `cbCut` = better architectural blueprints that prevent bad designs from ever being built. Fewer disasters to inspect later. (efficiency).

Separation Oracle for Fractional Solutions

Given a fractional solution $x^* \in [0, 1]^{|E|}$, we want to find a subset $S \subset V, S \neq \emptyset, S \neq V$ such that:

$$\sum_{i \in S, j \notin S} x^*_{ij} < 1$$

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So finding the **most violated** DFJ constraint = finding the **global minimum directed cut**.

If the minimum cut value < 1: that cut set S gives a violated DFJ inequality.
If the minimum cut value ≥ 1: all DFJ constraints are satisfied.

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What you can read:

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# Status of the node LP (must be OPTIMAL to read solution)
status = model.cbGet(GRB.Callback.MIPNODE_STATUS)

# The fractional LP solution (only if status == GRB.OPTIMAL)
xstar = model.cbGetNodeRel(model._x)

# Current node count
nodecnt = model.cbGet(GRB.Callback.MIPNODE_NODCNT)

# Current best objective bound
objbnd = model.cbGet(GRB.Callback.MIPNODE_OBJBND)
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```

What you can do:

```
# Add a globally valid inequality to tighten the relaxation
model.cbCut( <linear expression> >= <rhs> )
```

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Always guard your cbCut code

```
if where == GRB.Callback.MIPNODE:
    status = model.cbGet(GRB.Callback.MIPNODE_STATUS)
    if status != GRB.OPTIMAL:
        return    # nothing to separate, skip!
    xstar = model.cbGetNodeRel(model._x)
    # ... now safe to work with xstar ...
```

Practical Tip: Limit cbCut to Early Nodes

Adding user cuts at **every** node can slow the solver down (oracle overhead).

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    nodecnt = model.cbGet(GRB.Callback.MIPNODE_NODCNT)
    if nodecnt > 100:       # stop adding cuts deep in the tree
        return

    xstar = model.cbGetNodeRel(model._x)
    # ... separate and add cbCut ...
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Why? The root node LP benefits the most from tightening (all subtree bounds improve). Deep nodes have small subproblems where the overhead is not worth it.

Fractional Separation: Global Min-Cut with igraph

```python
import igraph as ig  # networkx does not have global directed min-cut!

def mincut_separate_dfj(n, xstar):
    """
    Find the most violated DFJ constraint in a fractional solution.
    Returns (S, cut_value) if violated, else (None, None).
    """
    edges = [(i, j) for i in range(n) for j in range(n) if i != j]
    caps  = [float(xstar[i, j]) for (i, j) in edges]

    g = ig.Graph(n=n, edges=edges, directed=True)
    g.es["cap"] = caps

    cut = g.mincut(capacity="cap")   # global directed min-cut
    val = float(cut.value)

    if val < 1 - 1e-6:   # tolerance for numerical safety
        A, B = cut.partition
        S = set(A) if 0 < len(A) < n else set(B)
        if 0 < len(S) < n:
            return S, val
    return None, None
```

The Unified Callback

```python
def callback(model, where):
    x, n = model._x, model._n

    # (A) CORRECTNESS: reject integer solutions with subtours
    if where == GRB.Callback.MIPSOL:
        xsol = model.cbGetSolution(x)
        subtours = find_subtours(n, xsol)
        for S in subtours:
            complement = set(range(n)) - S
            model.cbLazy(
                gp.quicksum(x[i,j] for i in S for j in complement) >= 1)

    # (B) PERFORMANCE: tighten LP relaxation with DFJ cuts
    if where == GRB.Callback.MIPNODE:
        if model.cbGet(GRB.Callback.MIPNODE_STATUS) != GRB.OPTIMAL: return
        if model.cbGet(GRB.Callback.MIPNODE_NODCNT) != 0: return
        xstar = model.cbGetNodeRel(x)
        S, val = mincut_separate_dfj(n, xstar)
        if S is not None:
            comp = set(range(n)) - S
            model.cbCut(
                gp.quicksum(x[i,j] for i in S for j in comp) >= 1)
```

Putting It All Together

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- At fractional node LPs: `cbCut` tightens the relaxation $\Rightarrow$ better bounds, more pruning.
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- Result: correct answer, found faster.

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Rule

- Constraints required for **correctness** $\Rightarrow$ `cbLazy`.
- Constraints added only for **speed** $\Rightarrow$ `cbCut`.
- For TSP: we need `cbLazy` at minimum. `cbCut` is a performance bonus (and we should measure to see if it is actually helpful to the solver).

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Option 3: Add `cbCut` on top.

- You also have a separation oracle for **fractional** solutions.
- Purely a performance optimization, tightens the LP relaxation.

Hot Start for MIPs: Not as Magical as for LPs

In Lecture 8, we saw that LP row generation benefits enormously from **hot starts**:

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If you add a constraint and re-solve from scratch, **all of this is lost**. There is no "dual simplex trick" that cheaply repairs a B&B tree.

Why Callbacks Beat an Outer Loop for MIPs

Naive approach (outer loop):

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2. Check solution. If violated, add constraint.
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Key insight

For LPs, an outer loop with hot starts works great (Lecture 8).

Common Pitfalls

1. Forgetting `LazyConstraints = 1`

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3. Numerical tolerances

```
# BAD: exact comparison with floating point
if cut_value < 1:  ...

# GOOD: use a small tolerance
if cut_value < 1 - 1e-6:  ...
```

Common Pitfalls (continued)

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# Practical pattern: limit cuts to early nodes
if where == GRB.Callback.MIPNODE:
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5. Using `cbCut` alone for correctness
Gurobi treats user cuts as **hints**. It may discard them during presolve or node processing. Never rely on `cbCut` for constraints that are required for a correct answer, use `cbLazy` for those.

Useful Callback Attributes

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These are useful for logging, early termination, or deciding when to stop adding cuts.